Chapter

An Alternative Approach to Probability in Quantum Information Science

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Abstract

This chapter provides a probabilistic framework for formulating classical probability theory, quantum probability, thermodynamics, diffusion, and the Wiener integral using a set of four axioms or principles. It explains everything that conventional quantum information theory and classical probability theory achieve. We want to emphasize that this framework is not an interpretation of quantum mechanics such as "Many-Worlds," "Bohm's Theory," or the "Copenhagen interpretation." It is much more general and can be viewed as a probability algorithm that calculates probabilities of future events. As a result, previously perplexing paradoxes find resolution. In particular, the superposition principle takes on a new meaning. Our probabilistic framework stands apart from the Hilbert space formalism. It relies solely on elementary set theory, classical logic, and complex numbers. Consequently, this theory is accessible for instruction in educational settings. This framework can be regarded as an axiomatic approach to probability in the sense of Hilbert. In his sixth of the twenty-three open problems presented at the International Congress of Mathematicians in Paris in 1900, Hilbert called for an axiomatic probability theory.

Keywords: quantum information theory, quantum physics, probability axioms, Feynman's formulation, thermodynamics, diffusion, Brownian motion

1. Introduction

> *The true logic of the world is in the calculus of probabilities.* James Clerk Maxwell

According to the Cambridge dictionary, probability is a nonnegative number representing how likely a particular outcome in a random experiment will happen. David Hilbert introduced his renowned set of fundamental problems, including the sixth problem, which aimed to establish an axiomatic foundation for probability theory, much like geometry. Several notable responses to Hilbert's challenge have resurfaced since then, see [1]. However, it is worth noting that more than a century later, von Weizsäcker ([2], p. 59), further delved into this topic:

> *Probability is one of the outstanding examples of the epistemological paradox that we can successfully use our basic concepts without actually understanding them.* von Weizsäcker [2]

Even today, classical probability, its axioms, and how to assign probabilities to elementary events is a philosophical dispute discussion. There are various interpretations of probability, including one of the oldest, the frequency interpretation. See [1, 3], and the literature therein for different interpretations and axioms.

The concepts of information and probability are closely linked. For instance, the Shannon concept of information relies on probability, such as the thermodynamic concept of entropy. In the same way, quantum states are probabilistic states of information.

About the probability in quantum information science, Weinberg [4] writes:

> *Even so, I'm not as sure as I once was about the future of quantum mechanics. It is a bad sign that those physicists today who are most comfortable with quantum mechanics do not agree with one another about what it all means. The dispute arises chiefly regarding the nature of measurement in quantum mechanics.* Weinberg [4]

Regarding quantum probability problems, discussions often take on a strange and peculiar attitude, as Fuchs [5] noted in his reflections on the annual conferences.

> *What is the cause of this year-after-year sacrifice to the "great mystery?" Whatever it is, it cannot be for want of a self-ordained solution: Go to any meeting, and it is like being in a holy city in great tumult. You will find all the religions with all their priests pitted in holy war — the Bohmians* [3], *the Consistent Historians* [4], *the Transactionalists* [5], *the Spontaneous Collapseans* [6], *the Einselectionists* [7], *the Contextual Objectivists* [8], *the outright Everettics* [9, 10], *and many more beyond that. They all declare to see the light, the ultimate light. Each tells us that if we accept their solution as our savior, then we too will see the light.* Fuchs [5]

A detailed description of interpretations of quantum mechanics, including many references, can be found in [6]. We also mention the easily readable WIKIPEDIA article "interpretations of quantum mechanics."

In this chapter, we introduce a predictive algorithm designed for calculating probabilities concerning future macroscopic events or detector clicks. Our principles maintain a strict separation of internal possibilities and outcomes, leading to a broader scope than the axioms of quantum mechanics. This chapter summarizes some essential parts of three lecture notes [7–10], including some corrections.

The chapter is organized as follows. The basic axioms of classical probability and the fundamental add and multiply rule, meaning that "probabilities for disjoint events are added, and probabilities for independent events are multiplied," are introduced in Section 2. Moreover, it is emphasized that classical probability and quantum probability are not compatible. The main topic of this chapter is a probabilistic framework, consisting of four general principles, which, in particular, allow the reconstruction of classical probability *and* quantum probability. These principles form the content of Section 3. Several examples, including the double-slit experiment, are presented in Section 4. Some more details about Hilbert's sixth problem are given in Section 5. In Section 6, we show that our probabilistic framework is consistent and contains a $U(1)$ symmetry as in quantum electrodynamics. In Section 7, we present the reconstruction of Feynman's formulation of quantum mechanics, which is mathematically equivalent to Schrödinger's theory and Heisenberg's matrix mechanics. Its close relationships to Brownian motion, Wiener integrals, and diffusion are described in Section 8. The reconstruction of statistical thermodynamics, described in Section 9, is a vital touchstone when applying a probabilistic theory. Finally, some conclusions are given.

2. The Kolmogorov axioms

In 1933, Kolmogorov presented a mathematical theory of classical probability in terms of some axioms that have since become standard. He uses elementary set theory. Given two sets A and B, the *union* $A \bigcup B$ denotes the set whose elements appear either in A or in B, or in both. The *intersection* $A \cap B$ denotes the set whose elements appear in both sets A and B. If A is a subset of B, then the *complement* A^c is the subset of B whose elements are not in A.

Kolmogorov's axioms are based on two fundamental sets:

i. The *sample space* $\mathbf{O}$ of outcomes. The outcomes are also called *elementary events*.

ii. The *probability algebra* $\mathbf{A}$ of subsets of $\mathbf{O}$ that contains $\mathbf{O}$ itself and is closed under complement and countable unions. Sometimes, this algebra is called *field*. The subsets $A \in \mathbf{A}$ are called *events*.

    Moreover, there exists a mapping Pr, called *probability* from the field of events $A \in \mathbf{A}$ into the set of nonnegative numbers:

iii.
$$A \to \Pr(A), \quad 0 \leq \Pr(A) \leq 1, \quad \Pr(\mathbf{O}) = 1, \tag{1}$$

and for any countable set of disjoint events A_m, the equation

$$\Pr\left(\bigcup_{m=1}^{\infty} A_m\right) = \sum_{m=1}^{\infty} \Pr(A_m). \tag{2}$$

must be fulfilled.

The condition that probabilities are numbers between zero and one is essential since otherwise, we cannot hope that the relative frequencies of an event A approach PR(A). The *relative frequency* is the number of times the event A occurred in a series of executions of an experiment divided by the number of executions, thus being bounded between zero and one.

Two events A and B are called *independent* if both do not influence each other. For instance, if we toss a coin twice and know the outcome A of the first toss, then this has no influence on the result B of the second toss. The probabilities for independent events are multiplied, that is,

$$\Pr(A \cap B) = \Pr(A)\Pr(B). \tag{3}$$

In summary, the probabilities of disjoint events are added, and the probabilities of independent events are multiplied. This is the well-known *multiply and add rule*, which holds valid already for Laplace experiments.

The mathematics of Kolmogov's probability theory is well understood, but its interpretation is controversial, see also [3].

It is well-known that classical probability and quantum probability are incompatible and contradictory. For example, Anthony Zee [11] writes on page 141 under the title "Dice Unlike Any Dice:"

> *Welcome to the strange world of the quantum, where one cannot determine how a particle gets from here to here. [...] When a die is thrown, the probability of getting a 1 is 1/6. The probability of getting a 2 is, of course, also 1/6. Now, consider the following question: What is the probability of getting a 1 or a 2 in one throw? The answer is evident to gamblers and non-gamblers alike: The probability is 1/6 + 1/6 =1/3. In everyday life, to obtain the probability of either A or B occurring, we simply add the probability of A occurring and the probability of B occurring.*
>
> *The quantum die is astonishingly different. Suppose we are told that for the quantum die the probability of throwing a 1 is 1/6, and the probability of throwing a 2 is also 1/6. In contrast to what our experience with ordinary dice might suggest, we cannot conclude that the probability of getting either a 1 or a 2 in one throw is 1/3! It turns out that the probability of throwing a 1 or a 2 can range between 1/3 and 0!*

Apparently, quantum theory yields results other than classical probability theory, and the question arises about a more fundamental theory below both theories.

The main aim of this publication is to present a general probability theory that simultaneously allows the treatment of classical stochastic and quantum mechanical experiments.

3. A unified probabilistic framework

An opinion on quantum mechanics, held by numerous physicists, is eloquently articulated in the book authored by Susskind and Friedman ([12], p. 24):

> *For a classical system, the space of states is a set (the set of possible states), and the logic of classical physics is Boolean. That seems obvious, and it is not easy to imagine any other possibility. Nevertheless, the real world operates along different lines, at least whenever quantum mechanics is important. The space of states of a quantum system is not a mathematical set [6]; it is a vector space. Relations between the elements of a vector space are different from those between the elements of a set, and the logic of propositions is different as well.*

Is a Hilbert space formalism and a modified logic indispensable in quantum physics? We describe a probabilistic framework that unifies classical mechanics, statistical thermodynamics, and quantum mechanics, not based on Hilbert spaces but on classical logic. It is based on decidable alternatives, which we call *outcomes*. The outcomes are described by sets consisting of *elementary possibilities*. Our approach partially supports the opinion of Fuchs and Peres [13]:

> *The thread common to all the nonstandard "interpretations" is the desire to create a new theory with features corresponding to some reality independent of our potential experiments. But, trying to fulfill a classical worldview by encumbering quantum*

> *mechanics with hidden variables, multiple worlds, consistency rules, or spontaneous collapse without any improvement in its predictive power only gives the illusion of a better understanding. Contrary to those desires, quantum theory does not describe physical reality. What it does is provide an algorithm for computing probabilities for the macroscopic events ("detector clicks") that are the consequences of our experimental interventions. This strict definition of the scope of quantum theory is the only interpretation ever needed, whether by experimenters or theorists.* Fuchs and Peres [13]

In the following, we consider random experiments in the broadest sense. They are described by three sets:

i. The *possibility space* $\mathbf{P}$ is a set with elements $p \in \mathbf{P}$. We call its elements *elementary possibilities*.

ii. The *possibility algebra* $\mathbf{F}$ is defined as the collection of subsets of $\mathbf{P}$ that contains $\mathbf{P}$ itself and is closed under complement and countable unions, that is, $\mathbf{F}$ is a *field*. The subsets $F \in \mathbf{F}$ are called *possibilities*. If F does not correspond to an elementary possibilities $\{p\}$, then F is called *nonelementary*.

iii. The *sample space* $\mathbf{O}$ consists of pairwise disjoint sets $F \in \mathbf{F}$ called *outcomes*. The outcomes form a partition of the possibility space, that is, each elementary possibility $p \in \mathbf{P}$ is contained in exactly one outcome F. If an outcome F consists of more than one element, we call its elements *internal elementary possibilities*, which are *accessible* from F.

   Additionally, we demand the existence of a function that evaluates possibilities:

iv. A *probability amplitude* is defined as a mapping φ from the possibility algebra $\mathbf{F}$ into the set of complex numbers:

$$F \to \varphi_F = \varphi(F) \in \mathbb{C}, \; F \in \mathbf{F}. \tag{4}$$

We claim that these quantities satisfy two general principles or axioms.

First principle: Given a countable set of pairwise disjoint possibilities $F_m \in \mathbf{F}$, in order that $F = \bigcup_m F_m$, it is

$$\varphi_F = \varphi\left(\bigcup_m F_m\right) = \sum_m \varphi_{F_m}. \tag{5}$$

This principle is the *superposition of probability amplitudes*. It is very general compared to Feynman's first principle: "When an event can occur in several alternative ways, the probability amplitude for the event is the sum of the probability amplitudes for each way considered separately" ([14], pp. 1–16). Feynman's quantum mechanics does not distinguish between outcomes, possibilities, and internal possibilities. Hence, it differs from our framework. Moreover, Feynman uses the four-dimensional space-time, and we require only the partitioning future, present, and past.

Second principle: This is *Born's rule*. It transforms probability amplitudes of outcomes F to probabilities $Pr(F)$:

$$\Pr(F) = \ |\varphi_F|^2 \ \text{ for all } \ F \in \mathbf{O}, \ \text{ and } \sum_{F \in \mathbf{O}} |\varphi_F|^2 = 1. \tag{6}$$

Born's rule states that the probability of measuring a specific outcome F is proportional to the square of the absolute value of the probability amplitude associated with that outcome. Summing up the probabilities of all outcomes, we get one, such that Born's rule implies a real probability measure on the sample space $\mathbf{O}$. In particular, classical probability is incorporated.

We call the quadruplet $(\mathbf{P}, \mathbf{F}, \mathbf{O}, \varphi)$, together with these two principles, a *possibility measure space*.

It is worth noting that, in the literature, a measure is often defined as a nonnegative function, in contrast to the use of complex amplitudes here. Nevertheless, it is crucial to emphasize that complex numbers are both indispensable and fundamental for accurately describing quantum physical reality, as supported by several references: [12], page 44, [7], Section 2.2, [15].

The probabilities for the outcomes belong to the prognostic category future. In the category present, the experiment is performed. For example, a particle runs through the experimental setup. This particle has no idea of the experimental setup and the placed detectors. The only thing it does is to act according to the probabilities: There is no rest, and the particle tends to move toward states of larger probability. Notice that this point of view is fundamentally different compared to the widely celebrated wave-particle duality.

These two principles are mathematical conditions that must be satisfied for probability amplitudes. The first principle implies that it is sufficient to compute the amplitudes for the elementary possibilities only. The second one, Born's rule, says we must calculate only the amplitudes for the outcomes. Now, we introduce two further principles that help to compute concrete probability amplitudes.

Third principle: The amplitudes φ_F contribute equally in magnitude for all accessible elementary possibilities. They are proportional to some constant times a complex number of magnitude one, namely

$$e^{\frac{i}{\hbar}S(F)}. \tag{7}$$

The function $S(F)$ is called the *action* of the elementary possibility F.

Feynman's formulation of quantum theory is as follows:

> *The total amplitude can be written as the sum of amplitudes of each path - for each way of arrival. For every $x(t)$ that we could have - for every possible imaginary trajectory - we have to calculate an amplitude. Then, we add them all together. What do we take for the amplitude of each path? Our action integral tells us what the amplitude for a single path ought to be. The amplitude is proportional to some constant times $\exp(iS/\hbar)$, where S is the action for the path. If we represent the phase of the amplitude by a complex number, Planck's constant $\hbar$ has the same dimensions.*
> Feynman and Hibbs [16], page 19.

In our third principle, no further requirements are made about the action, as is necessary in the case of space-time paths. Consequently, this principle is very flexible in describing physical problems outside space-time.

In classical probability theory, the Laplace *principle of indifference* says that all outcomes are equally likely assigned with unit one. Hence, the difference is that we merely replace unit one with complex numbers of magnitude one. It follows that we get back the theory of Laplace if we set the phase equal to zero.

Fourth principle: We call two possibilities F and G *independent* provided their intersection is non-empty, and the occurrence of one possibility does not affect the other one. Mathematically, independence is defined by the equation:

$$\varphi_{F\cap G} = \varphi_F \ \varphi_G. \tag{8}$$

In other words, the joint amplitude is equal to the product of their amplitudes.

Feynman ([14], pp. 3–4), describes independence as follows: "When a particle goes by some particular route, the amplitude for that route can be written as the product of the amplitude to go partway with the amplitude to go the rest of the way." Independence is a fundamental concept in probability theory and statistics because it simplifies calculations and allows for modeling complicated random experiments. Laplace already introduced it in the late eighteenth and early nineteenth centuries. It says that an experiment, which breaks down into a series of possibilities happening independently, the probability of the occurrence of all possibilities is the product of the probability of each. Our first and fourth principle shows that the well-known *multiply and add rule* in probability theory carries over to complex probability amplitudes for possibilities.

The physical content of this theory lies in the third principle *via* the classical action $S(F)$. In contrast, the other principles are purely mathematical and physically empty. Notice that the stationary-action principle is a variational principle, yielding the equations of motion in Newtonian mechanics, general relativity, and classical electrodynamics when applied to the corresponding action. For example, in general relativity, it is the Einstein-Hilbert action. In quantum field theory, the action is incorporated into the path integral.

4. Examples

4.1 Tossing a die

A simple example is tossing a fair die. There are six elementary possibilities $1, 2, 3, 4, 5, 6$ yielding the possibility space $\mathbf{P} = \{1, 2, 3, 4, 5, 6\}$. The corresponding possibility algebra $\mathbf{F}$ is the power set of $\mathbf{P}$. The six outcomes correspond to the six elementary possibilities. They form a partition of the possibility space. We define the action as equal to zero. Then, the exponential interference term in Eq. (7) is equal to one. We set

$$\varphi_\varnothing = 0, \ \ \varphi_{\{k\}} = \frac{1}{\sqrt{6}}, \ \ \text{for all} \ \ k \in \mathbf{P}. \tag{9}$$

Hence, the probabilities for all outcomes are $1/6$, according to the second principle. The first principle yields

$$\varphi_{\{\mathbf{P}\}} = 6 \times \frac{1}{\sqrt{6}} = \sqrt{6}. \tag{10}$$

The value $\left|\varphi_{\{\mathbf{P}\}}\right|^2 > 1$ is no contradiction because Born's rule is only applied to the outcomes, not to arbitrary possibilities. Notice that the probabilities of the outcomes satisfy Kolmogorov's axioms.

The probability amplitudes and the related probabilities are prognostic numbers for future events. If we execute an experiment in the present, the results agree with these probabilities.

4.2 Atom in two states

In his book, Smolin [17], Chapter 4, presents a simple quantum experiment of an atom that can exist in two states: an excited state denoted as E and a ground state with the lowest energy, denoted as G. The atom, while in the unstable excited state E, has the capability to transit to the ground state G by emitting a photon. To explore this, we place an excited atom inside a sealed box alongside a Geiger counter. Much like the atom, the Geiger counter has two possible states: the yes-state Y, meaning that it has detected a photon, and the no-state N indicating that no photon has been detected.

Initially, the system is in the state (E, N) with the atom in the excited state and the Geiger counter in the no state. After a certain duration, when we open the box, we find that the system is in one of two possible states: Either it remains in the initial state (E, N) or it has transitioned to the state (G, Y) with the atom in the ground state and the Geiger counter in the yes state.

The postulates of quantum mechanics tell us that, before opening the box, the system is in a *superposition* of both states

$$(E, N) \ \ \textbf{and} \ \ (N, Y). \tag{11}$$

Smolin calls this the "in-between" state. However, we have never observed a superposition after opening the box. This is a seemingly weird situation, as Smolin writes. Then, he raises some questions. Why has quantum mechanics two dynamical rules, the *unitary evolution* before opening the box, and the *collapse* into one of the states (E, N) *or* (G, Y) when opening the box? This is in contrast to other theories that only have one dynamic. Why does the process of measurement differ from other processes? When does the collapse occur? Does it happen when the particle interacts with the counter or when the box is opened, and an observer becomes conscious of the outcome? These and other questions are typical in quantum mechanics.

Our framework is purely probabilistic, calculating numbers of future events. In the present, when performing an experiment, the detector clicks agree with the calculated probabilities in the sense of relative frequencies. Questions as above do not occur. For the atom with two states and the Geiger counter, the possibility space has the form

$$\mathbf{P} = \{(E, N), (E, Y), (G, N), (G, Y)\}. \tag{12}$$

The related possibility algebra is the power set of $\mathbf{P}$. The outcomes coincide with the elementary possibilities. In other words, the sample space of outcomes and the possibility space are identical if we identify $p \in \mathbf{P}$ with $\{p\} \in \mathbf{F}$. There exist no internal possibilities. We have a simple classical statistical situation.

We set

$$\varphi(E,Y) = \varphi(G,N) = 0, \ \ \varphi(E,N) \neq 0, \ \ \varphi(G,Y) \neq 0. \tag{13}$$

All quantities belong to the prognostic future, that is, they describe what might happen. The nonzero amplitudes depend on the experimental setup, the type of atoms, and how long the particle is in the closed box. Born's rule tells us that

$$|\varphi(E,N)|^2 + |\varphi(G,Y)|^2 = 1. \tag{14}$$

In the present, the experimental results show that the atom tends to move to higher probability outcomes. Nothing is strange; we require no "in-between" superpositions.

Smolin's example is directly related to the well-known Schrödinger's cat thought experiment, a famous quantum mechanical illustration devised by Schrödinger in 1935. It is designed to highlight the concept of superposition or "in-between" states and the peculiar nature of quantum mechanics. In the experiment, a hypothetical cat is placed in a sealed box with a radioactive atom, a Geiger counter, a vial of poison, and a mechanism that will release the poison if the Geiger counter detects radiation. According to the principles of quantum mechanics, before the box is opened and the cat is observed, the cat's state is in between alive and dead. In other words, until the box is opened, the cat is considered alive and dead simultaneously. In our framework, the cat is either dead or alive. Nothing strange happens.

4.3 The double-slit experiment

The double-slit experiment has been called "the most beautiful experiment in physics" [18]. Frequently, its interpretation is that particles of matter behave like a wave and that the act of observing a particle has a dramatic effect on the experimental results. It can be performed with photons or electrons. Actually, experiments with large molecules composed of more than 800 atoms indeed show interference. In 2012, physicists from Vienna used large molecules called phthalocyanine, which can be seen with a video camera exhibiting their macroscopic nature. The experiment is executed such that only one molecule interacts with the setup. They arrive localized at small places at the final wall of detectors, a behavior typical for macroscopic objects, not for classical waves.

Imagine a source producing particles. Behind is a wall with two slits in it and, after that, a screen of detectors. If we execute the experiment, some particles will bounce off the wall, but some will travel through the slits and will arrive at the screen, see **Figure 1**. We consider only the particles arriving at the screen. We define the elementary possibilities as follows: They are piecewise straight paths sad_m, sbd_m from the source s, *via* the wall W with two slits a and b, to the detectors $d_m, m = -l, \ldots, l$ positioned on the screen D.

We consider three experimental setups. Firstly, only one slit is open. Secondly, both slits are open, and thirdly detectors measure through which slit the particle goes. We shall see how these changes in the experimental setup change the statistics significantly.

Firstly, let slit b be closed, that is, only paths through slit a are relevant. Hence, the possibility space is

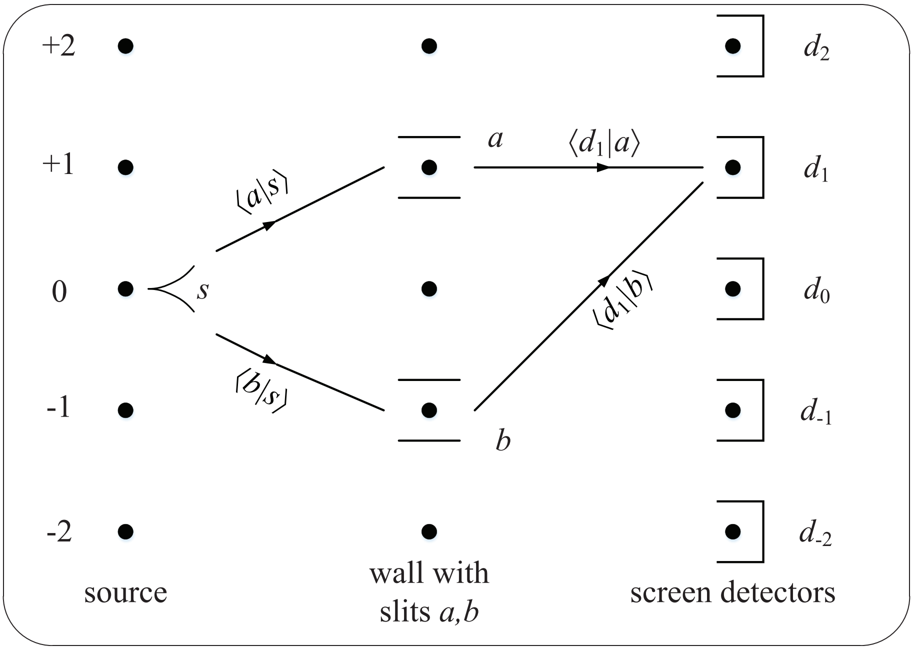

Figure 1.
The double-slit experiment: a particle leaves source s, passes one of the two slits a or b, and is detected in d_1, finally.

$$\mathbf{P} = \{sad_m : \ d_m \in D\}. \tag{15}$$

We have no internal possibilities such that the sample space of outcomes

$$\mathbf{O} = \{O_{d_m} : \ d_m \in D\}, \ \ O_{d_m} = \{sad_m\} \in \mathbf{F} \tag{16}$$

corresponds uniquely to **P**. This is a classical experiment. Our third principle yields the amplitude

$$\varphi(O_{d_m}) = \varphi_{sad_m} \tag{17}$$

via the action of the path sad_m. The squared magnitudes of the amplitudes are the probabilities:

$$\Pr(O_{d_m}) = \ \left|\varphi_{sad_m}\right|^2 \tag{18}$$

In the same way, we obtain the probability

$$\Pr(O_{d_m}) = \ \left|\varphi_{sbd_m}\right|^2, \tag{19}$$

when slit a is closed.

Now secondly, we suppose that both slits are open. Then the possibility space consists of all paths from the source to the detectors

$$\mathbf{P} = \{sad_m, sbd_m : \ a, b \in W, d_m \in D\}. \tag{20}$$

We have internal possibilities since it cannot be observed through which slit the particle goes in the present. The sample space of outcomes is defined as

$$\mathbf{O} = \left\{O_{d_m} : \; d_m \in D\right\}, \quad \text{where} \;\; O_{d_m} = \{sad_m, sbd_m\}. \tag{21}$$

Using the third principle, we set

$$\varphi_{sad_m} = \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar}S(sadm)}, \;\; \varphi_{sbd_m} = \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar}S(sbdm)}, \tag{22}$$

and the first principle yields the amplitudes of the outcomes

$$\varphi\left(\left\{O_{d_m}\right\}\right) = \varphi_{sad_m} + \varphi_{sbd_m} \;\; \text{for all} \;\; d_m \in D. \tag{23}$$

Born's rule provides the probabilities of the outcomes:

$$\begin{aligned} \Pr(O_{d_m}) &= \left| \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar}S(sadm)} + \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar}S(sbdm)} \right|^2 \\ &= \frac{1}{2}\left(\left| e^{\frac{i}{\hbar}S(sadm)} \right|^2 + \left| e^{\frac{i}{\hbar}S(sbdm)} \right|^2 \right) \\ &+ \frac{1}{2}\left(\left(e^{\frac{i}{\hbar}S(sadm)}\right)^* e^{\frac{i}{\hbar}S(sbdm)} + \left(e^{\frac{i}{\hbar}S(sbdm)}\right)^* e^{\frac{i}{\hbar}S(sadm)} \right). \end{aligned} \tag{24}$$

Compared with the case where one slit is closed, the first term in this sum corresponds to the classical probability. The second term is responsible for interference.

In the case where $e^{\frac{i}{\hbar}S(sadm)} = e^{\frac{i}{\hbar}S(sbdm)}$, Eq. (24) yields

$$\Pr(O_{d_m}) = 2\left| e^{\frac{i}{\hbar}S(sadm)} \right|^2. \tag{25}$$

Thus, the probability when only one slit is open is doubled, and we get *constructive interference*. For the other extreme case where $e^{\frac{i}{\hbar}S(sadm)} = -e^{\frac{i}{\hbar}S(sbdm)}$, we get the probability

$$\Pr(O_{d_m}) = 0, \tag{26}$$

yielding *destructive interference*.

We have computed only probabilities of future events, yielding a pattern of constructive and destructive interference. In the present, a particle chooses a path. Preferably, those with high probability.

Finally, suppose we have information about the slit where a particle passes. This information comes about by two additional detectors d_a and d_b at the slits. Assume that the detectors work correctly such that it cannot happen that a particle arrives at detector d_m *via* slit b and detector d_a clicks, or both detectors d_a and d_b do not click.

Then the possibility space is defined as

$$\mathbf{P} = \{sad_a d_m, sbd_b d_m : \quad a, b \in W, d_m \in D\}. \tag{27}$$

The outcomes are defined via the detector clicks at the screen and the clicks of two additional detectors d_a and d_b. Hence, we obtain the sample space of outcomes:

$$\mathbf{O} = \left\{O_{d_a d_m}, O_{d_b d_m} : \quad d_m \in D\right\}, \tag{28}$$

where

$$O_{d_a d_m} = \{sad_a d_m\}, O_{d_b d_m} = \{sbd_b d_m\}. \tag{29}$$

Using Born's rule, we get the classical probabilities:

$$\Pr(O_{d_a d_m}) = \left|\varphi_{sad_m}\right|^2, \quad \Pr\left(O_{d_b d_m}\right) = \left|\varphi_{sbd_m}\right|^2. \tag{30}$$

In summary, the numbers computed for the three different experimental setups are probabilities that describe how likely in the *future* a particle would meet one of the detectors. In the *present*, the particle does not know anything about the experimental setup. It passes the experiment with the tendency to move on exactly one path of higher probability. Of course, in rare cases, the particle will also choose paths with low probability. This natural explanation is all what we need to know. We see that it is essential to distinguish clearly between *elementary possibilities* and *outcomes*. Then interpretations, such as "wave-particle dualism," "many-worlds," "non-locality," and others are unnecessary. In particular, a material object does not occupy several locations at the same time as Penrose writes in his excellently written book [19] on page 216:

> *As we have seen, particularly in the previous chapter, the world actually does conspire to behave in a most fantastical way when examined at a tiny level at which quantum phenomena hold sway. A single material object can occupy several locations at the same time and like some vampire of fiction (able, at will, to transform between a bat and a man) can behave as a wave or as a particle seemingly as it chooses, its behavior is governed by mysterious numbers involving the "imaginary" square root of -1.* Penrose [19]

Penrose gave, not unfounded, his book the title FASHION, FAITH, and FANTASY. Our aim is, however, to show that the world is stochastic, at least their physical descriptions, but in no way fantastical and mysterious.

Large macroscopic molecules or other objects can be described as a cloud of elementary particles the constituents. Suppose the binding force between these constituents is very weak. Then the constituents in this cloud can independently of one another move through both slits yielding interference. But when the binding force between the constituents is large, then all move through the same slit. Then we get a stochastic pattern as in the case where only one slit is open.

It is easy to generalize the double-slit experiment to finitely many slits and to finitely many subsequent walls. Then the possibility space consists of all possible paths from the source *via* the walls to the detectors. Passing over to infinitely many walls with infinitely many slits leads to Feynman's path integral. For several other aspects of slit experiments, see Jansson [9], Chapter 4.

5. Hilbert's sixth problem

Hilbert's sixth problem [20] asks how to axiomatize those branches of physics in which mathematics, in the first rank the theory of probabilities, is prevalent. The aim is to treat physics by means of axioms, as in geometry.

Several different systems of axioms exist for probability theory. One of the most commonly used and well-known sets of axioms is the Kolmogorov axioms. These axioms are contained in our four principles *via* Born's rule for the outcomes. Our axiomatic approach to probability theory is similar to the axiomatic approach in geometry, where the foundational principles, such as Euclid's axioms, provide the basis for the development of geometric concepts and theorems. The axiomatic system in geometry consists of the following components:

- The primitives: points, lines, and planes.
- The axioms are statements about these primitives; for instance, two points are together incident with one line.
- The laws of logic.
- The theorems that are the logical consequences of the axioms.

According to Hilbert, primitive terms are empty shells or placeholders with no intrinsic properties. It means that instead of points, lines, and planes, we can also use the words windows, chairs, and houses. A concrete meaning of the primitives of a geometrical system yields a model of the axiomatic system, where all theorems are true statements in this model. Our possibility measure space may be viewed as an axiomatic probability theory in the sense of Hilbert's sixth problem, which is composed of the following components:

- The primitives: elementary possibilities, outcomes, and amplitudes.
- The axioms are statements about these primitives; for instance, each elementary possibility is contained in exactly one outcome.
- The laws of classical logic.
- The theorems, such as the inclusion-exclusion principle [9].

We shall consider several concrete models of our principles or axioms, thereunder Feynman's formulation of quantum mechanics in space-time, Wiener processes, and thermodynamics.

6. Consistency and symmetry

In the following, we prove the internal consistency of our probability theory, ensuring that it remains free from contradictions. Furthermore, we establish that our theory possesses a $U(1)$ symmetry, meaning that all probabilistic statements remain unchanged when the amplitudes associated with individual possibilities are transformed by a single element of the $U(1)$ group.

At first, we show that the probability amplitude φ_F is well-defined, that is, the amplitude should not depend on the partitioning of F. If F contains one element, there is nothing to prove. For two disjoint elements where $F = \bigcup\{F_1, F_2\}$, the amplitude

$\varphi_F = \varphi_{F_1} + \varphi_{F_2} = \varphi_{F_2} + \varphi_{F_1}$ is well-defined. In the case of three pairwise disjoint possibilities F_1, F_2, F_3, we partition $F = \bigcup\{F_1, F_2, F_3\}$ as follows:

$$F_1, F_2, F_3; \ \bigcup\{F_1, F_2\}, F_3; \ \bigcup\{F_1, F_3\}, F_2; \ \bigcup\{F_2, F_3\}, F_1. \tag{31}$$

Complex addition is associative and commutative. Hence, in each case, the first principle yields

$$\varphi_F = \varphi_{F_1} + \varphi_{F_2} + \varphi_{F_3}, \tag{32}$$

and φ_F is well-defined. Analogously, the same holds true when the partitioning consists of more than three elements:

$$\varphi_F = \sum_m \varphi_{F_m}. \tag{33}$$

The second principle, Born's rule, requires that the sum of the square of the magnitudes of all probability amplitudes that correspond to the outcomes is one. This is a simple normalization condition that can always be achieved.

Finally, due to Born's rule, multiplying all probability amplitudes with the same element $e^{i\phi} \in U(1)$ does not change the probabilities. Thus, our possibility measure space has a symmetry with respect to the fundamental symmetry group $U(1)$. It is well-known that quantum electrodynamics has a $U(1)$ gauge symmetry, justified by the fact that the absolute phase of the wave functions of electrons, photons, or other particles cannot be measured.

7. Reconstruction of quantum mechanics

In this section, we present a reconstruction of Feynman's quantum mechanics, rooted in the concept of path integrals. It is well-established that his theory is mathematical equivalent to both, Schrödinger's and Heisenberg's quantum formulations.

We introduce *Feynman's path integral* with the help of zigzag paths $x(t)$: Let a particle move from position x_a at time t_a to x_b at time t_b in space-time. The time is divided up into n smaller segments $t_a = t_0 < t_1 < \cdots < t_{n-1} < t_n = t_b$. All have the length $\varepsilon = (t_b - t_a)/n$.

The *possibility space* **P** contains finitely many zigzag paths from $a = (x_a, t_a)$ to $b = (x_b, t_b)$ where b varies in some subset B of the space-time. This subset may consist of various points where detectors are positioned. The *possibility algebra* **F** is the power set of **P**.

For fixed $b \in B$, the nonelementary possibility

$$F(b, a) = \{x(t) \in \mathbf{P} : x(t_a) = x_a, x(t_b) = x_b\} \in \mathbf{F} \tag{34}$$

defines an outcome. The sample space **O** consists of all sets $F(b, a)$ where b varies in B. They are pairwise disjoint and form a partitioning of **P**.

Let $c = (x_c, t_c) \in C$ be a space-time point such that $t_a < t_c < t_b$. We define the nonelementary possibility

$$F(b, c, a) = \{x(t) : x(t_a) = x_a, x(t_c) = x_c, x(t_b) = x_b\} \in \mathbf{F}. \tag{35}$$

Then

$$F(b, c, a) = F(b, c) \cap F(c, a), \tag{36}$$

where the sets on the right-hand side are defined as above. It follows that

$$F(b, a) = \bigcup_{c \in C} F(b, c, a). \tag{37}$$

Since the paths $x(t) \in \mathbf{P}$ are pairwise disjoint, the first principle implies Feynman's path integral:

$$\varphi(F(b, a)) = \sum_{x(t) \in F(b, a)} \varphi(\{x(t)\}). \tag{38}$$

The amplitude $\varphi(F(b, a))$ is well-known and also called *Green's kernel of motion*. Frequently, it is denoted by $K(b, a)$. Using Born's rule, we get the probability $Pr(b, a) = |K(b, a)|^2$ to move from a to b.

The *action* of a path is defined as the integral over its Lagrangian L

$$S(\{x(t)\}) = \int_{t_a}^{t_b} L(\dot{x}, x, t) dt. \tag{39}$$

We obtain the amplitude for the elementary possibilities with the third principle:

$$\varphi(\{x(t)\}) = \text{const} \quad \exp\left(\frac{i}{\hbar} S(\{x(t)\})\right), \tag{40}$$

We consider only zigzag paths. Thus, the action takes the form

$$S(\{x(t)\}) = \sum_{j=1}^{n} L\left(\frac{x_j - x_{j-1}}{\varepsilon}, \frac{x_j + x_{j-1}}{2}, \frac{t_j + t_{j-1}}{2}\right). \tag{41}$$

Formula (38) is the essence of the quantum formulation of Feynman. Now, we ask how to calculate the sum over all paths. We remember the Riemann integral of some function f, which is approximated in the form

$$\int_{x_a}^{x_b} f(x) dx \propto \sum_{j=0}^{n} f(x_j), \tag{42}$$

where the points x_j are equally spaced. This sum depends on the number n. In this form, a limit would not exist. But the normalization factor $\delta = (x_b - x_a)/n$ yields

$$\int_{x_a}^{x_b} f(x) dx = \lim_{\delta \to 0} \left(\delta \sum_{j=0}^{n} f(x_j)\right). \tag{43}$$

Similarly, we must introduce a normalization factor for the path integral. This is not trivial in concrete experiments.

Putting all together and taking the limit $\varepsilon = (t_b - t_a)/n \to 0$, *Feynman's path integral* Eq. (38) can be written as

$$K(b,a) = \lim_{\varepsilon \to 0} \frac{1}{A} \int \cdots \int \exp\left(\frac{i}{\hbar} S(\{x(t)\})\right) \; \frac{dx_1}{A} \; \cdots \frac{dx_{n-1}}{A}, \tag{44}$$

where A is a normalization constant depending on the Lagrangian.

The classical action is additive. Hence, Eq. (37), the first and fourth principle imply

$$S(b,a) = S(b,c) + S(c,a), \tag{45}$$

and it follows that

$$K(b,a) = \int_{x_c} K(b,c) K(c,a) dx_c. \tag{46}$$

More general, for $(n+1)$ points we get

$$K(b,a) = \int_{x_1} \int_{x_2} \cdots \int_{x_{n-1}} K(b, n-1) K(n-1, n-2) \cdots K(1,a) \; dx_1 dx_2 \cdots dx_{n-1} \tag{47}$$

where

$$K(j, j-1) = \frac{1}{A} \exp\left(\frac{i}{\hbar} \varepsilon L\left(\frac{x_j - x_{j-1}}{\varepsilon}, \frac{x_j + x_{j-1}}{2}, \frac{t_j + t_{j-1}}{2}\right)\right). \tag{48}$$

Now, we change the notation $x_b = x, t_b = t, x_a = y, t_a = s$. Then formula (46) can be written as a *wave function*, well-known in quantum theory:

$$\varphi(x,t) = \int K(x,t;y,s) \varphi(y,s) dy. \tag{49}$$

This formula says that the probability amplitude for the outcome of arriving at the point (x,t) is equal to the sum over all amplitudes to reach at (y,s) multiplied by the amplitude to move from (y,s) to (x,t).

In the prevailing formulation of quantum mechanics, the Schrödinger equation serves as the fundamental postulate. This equation can be derived from Eq. (49). We make an initial order approximation of the wave function with respect to the time interval ε, resulting in the formula:

$$\varphi(x, t+\varepsilon) = \frac{1}{A} \int \exp\left(\varepsilon \frac{i}{\hbar} L\left(\frac{x-y}{\varepsilon}, \frac{x+y}{2}, t\right)\right) \varphi(y,t) dy. \tag{50}$$

For example, in the special case of the Lagrangian $L = m\dot{x}^2 + V(x)$, we substitute $y = x + \mu$, integrate, and expand the resulting equation to first order in ε and second order in μ, yielding the *Schrödinger equation*

$$i\hbar \frac{\partial \varphi}{\partial t} = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi + V\varphi. \tag{51}$$

The corresponding normalization constant turns out to be (see [21], Section 6)

$$A = \sqrt{\frac{2\pi\hbar\varepsilon i}{m}}. \tag{52}$$

Using our probability theory, we successfully reconstructed Feynman's formulation based on path integrals, ultimately deriving the Schrödinger equation. The process of quantization naturally emerges from this equation, establishing it as a direct outcome of our probabilistic framework. Furthermore, it is worth noting that quantization can also be derived directly from the path integral, as demonstrated by Kleinert (cf. [21], Sections 2.6 and 9.2). In contrast, classical probability theory does not imply the concept of quantization.

It can be shown that, in the limit case, the paths may exhibit continuity but lack differentiability throughout space-time. In other words, the velocity is discontinuous at every point.

The phase space path integral offers a broader perspective compared to the space-time path integral discussed above. It introduces momentum as a crucial parameter, establishing a connection between quantum mechanics and the Hamiltonian formalism.

We will not provide a detailed derivation of this path integral formulation. For an in-depth exploration of path integrals, including comprehensive information and references, readers are encouraged to consult Kleinert's monograph [21] and the related literature. Additional insights on this topic can be found in the works of Feynman (cf. [14, 16, 22]).

8. Diffusion and Wiener Integral

Readers with knowledge of statistical mechanics will readily observe a striking resemblance between Feynman's formulation and the concept of Brownian motion, where discretization mirrors the behavior of discrete-time random walks. In fact, the path integral formulation is closely related to the mathematical framework of Brownian motion. In this section, we aim to briefly outline the connections between quantum path integrals, Brownian motion, diffusion processes, and the Wiener integral. For a more comprehensive exploration, readers are encouraged to consult Zeidler's book [23], Chapter 11 and explore the relevant literature.

The heat equation is defined as a partial differential equation, specifically addressing an initial value problem with an initial time parameter s:

$$\frac{\partial\varphi(x,t)}{\partial t} = -\kappa\frac{\partial^2}{\partial x^2}\varphi(x,t) - V(x)\varphi(x,t), \quad t \geq s, \quad \varphi(x,s) = \varphi_0(x). \tag{53}$$

In addition to its wide-ranging applications in scientific domains such as probability theory, financial mathematics, and image analysis, this equation provides a fundamental description of heat propagation within an isotropic and homogeneous medium. In this context, the variable $\varphi(x,t)$ represents the temperature at a specific spatial point, denoted by x, and at a particular moment in time, by t. Furthermore, this equation serves as a diffusion equation when applied to a mass density, with $\varphi(x,t)$ representing this density. At the microscopic level, diffusion is intimately connected to Brownian motion, which characterizes the stochastic and random movement of microscopic particles within gases or liquids.

In the book of Zeidler [23], Section 11.8, it is proved that its solution is

$$\varphi(x,t) = \int K(x,t;y,s)\varphi_0(y)dy, \tag{54}$$

where the heat kernel has the form

$$K(x,t;y,s) = \lim_{\varepsilon\to 0} \frac{1}{A}\int \cdots \int \exp(-S(\{x(t)\})) \; \frac{dx_1}{A} \; \cdots \frac{dx_{n-1}}{A}. \tag{55}$$

The symbol S represents the discrete action associated with a linear zigzag path, denoted as $x(t) = (x(t_i))$. For a Lagrangian, which is defined as the difference between the kinetic energy and the potential energy V, we have the expression:

$$S(\{x(t)\}) = \sum_{j=1}^{n} \frac{1}{4\kappa}\left(\frac{x_j - x_{j-1}}{\varepsilon}\right)^2 + V\left(x_j\right)\varepsilon. \tag{56}$$

We take the same discretization as in Section 7. The resulting normalization constant for points in the three-dimensional position space is

$$A = (4\pi\kappa\varepsilon)^{3/2}. \tag{57}$$

Much like Feynman's path integral, the heat kernel denoted as $K(x,t;y,s)$ represents the summation over all possible paths connecting the initial point y to the final point x.

The path integral in Eq. (55), which is also known as a *Wiener integral*, possesses a well-defined and rigorous interpretation as a classical measure within the realm of continuous functions ([24], Vol. II, Section X.11).

9. Thermodynamics

It is an important touchstone to reconstruct thermodynamics using our probability framework. For a more in-depth exploration of this reconstruction, we refer to Jansson ([9], Chapter 5). For those seeking a comprehensive introduction to the theory of thermodynamics, we recommend Penrose ([25], Chapter 27), and Ben-Naim [26, 27].

In thermodynamics, we often deal with an enormous number of constituents. To illustrate, just one mole of molecules corresponds to Avogadro's number, which is approximately on the order of 10^{23}. Consequently, thermodynamics fundamentally operates as a statistical theory.

The large collections of constituents are described in terms of *microstates*, where each constituent possesses attributes such as position, momentum, or energy. A microstate represents a specific configuration of a system where all microscopic variables are precisely determined. Microstates are distinct possibilities; they either occur or do not in the present, but two or more microstates cannot coexist simultaneously.

Macrostates, on the other hand, pertain to the overall thermodynamic system. These macrostates are characterized by a small set of macroscopic variables, such as the total energy E, pressure P, volume V, temperature T, or the total number N of molecules. Throughout the following discussion, we will use M to denote a macrostate and μ to denote a microstate.

The number of microstates, each representing precise configurations with exact microscopic values, can be immensely large. In contrast, a macrostate is defined by the fixation of a small number of macroscopic variables. Each macrostate encompasses a multitude of microstates, often referred to as *accessible* microstates. The *multiplicity* of a given *macrostate* denoted as M is the number of its accessible microstates and is represented as $\Omega(M)$. The *total multiplicity*, denoted as Ω_{tot}, is the sum of all the multiplicities $\Omega(M)$.

Macrostates are measurable in contrast to microstates, and they effectively partition the set of all microstates within the system.

The foundational principle of statistical thermodynamics asserts that *all microstates within a system are equiprobable*. As a consequence of this principle, the probability associated with a macrostate M is determined by the ratio of the multiplicity of that macrostate to the total multiplicity:

$$Pr(M) = \frac{\Omega(M)}{\Omega_{tot}}. \tag{58}$$

The obvious way to merge statistical thermodynamics with our probability framework is to identify the microstates μ as the elementary possibilities $p \in \mathbf{P}$ and to associate the macrostates M, as measurable states, to the outcome $F \in \mathbf{O}$.

Now, we can reevaluate the probabilities associated with macrostates Eq. (58) using our probabilistic framework. Our *third principle* posits that all elementary possibilities contribute equally in magnitude, meaning that the microstates μ can be expressed with amplitudes as follows:

$$\varphi_\mu = \text{const} \;\; e^{\frac{i}{\hbar}S(\mu)}. \tag{59}$$

Without further knowledge about the actions of the constituents, it is reasonable to assume that the action $S(\mu)$ is uniformly zero for all microstates. This choice renders the exponential term equal to one, indicating no interaction or interference. Furthermore, we set:

$$\text{const} = \frac{1}{\sqrt{\Omega_{tot}}\sqrt{\Omega(M)}}. \tag{60}$$

Then

$$\varphi_\mu = \frac{1}{\sqrt{\Omega_{tot}}\sqrt{\Omega(M)}} \cdot 1. \tag{61}$$

Since the microstates are mutually exclusive, we can invoke the first principle. Consequently, the probability amplitude of a macrostate M takes the form:

$$\varphi_M = \sum_{\mu \in M} \varphi_\mu = \Omega(M) \frac{1}{\sqrt{\Omega_{tot}}\sqrt{\Omega(M)}} = \sqrt{\frac{\Omega(M)}{\Omega_{tot}}}. \tag{62}$$

Following Born's rule, we obtain the classical probabilities for the outcomes as expressed in Eq. (58) when we compute the squared magnitude of probability amplitudes.

It is important to note that in our derivation, we did not employ the thermodynamic principle of indifference. Rather, our approach hinges on setting the action of all elementary possibilities (microstates) to zero. This choice pertains to the experimental setup rather than making a statement about probabilities.

Einstein writes about thermodynamics:

> *A theory is the more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability. Therefore the deep impression which classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that within the framework of the applicability of its basic concepts, it will never be overthrown.*
> Albert Einstein, autobiographical notes (1946)

with our probabilistic framework, we have covered many applications and reconstructed theories in addition to statistical thermodynamics.

We have introduced a probability theory describing future events. The future is timeless. Not surprisingly, the foundational theory of statistical thermodynamics, almost universally applicable, is also inherently timeless, as highlighted by the work of Ben-Naim [26]. The second law of thermodynamics and the concept of entropy are independent of time. This perspective aligns with the notion of "physics without time" advocated by some physicists, see Rovelli [28].

10. Conclusion

John Wheeler, as mentioned by Ballentine [29], contended that true comprehension of quantum theory demands the ability to encapsulate it within a single, readily understandable statement. Our succinct statement is as follows:

> *Quantum theory can be reconstructed through a simple probability framework that characterizes future events in terms of possibilities and outcomes, employing classical logic, straightforward set theory, and complex numbers*.

Our approach to quantum theory departs from conventional quantum mechanics in several key aspects. Moreover, our framework allows for the reconstruction of classical statistical mechanics and thermodynamics.

The theory appears to be simple enough to be taught even in schools, similar to Kolmogorov's theory of probability.

Theories and interpretations can significantly influence techniques and engineering practices. Quantum information theory provides insights into communication systems, data compression, and cryptography, essential in modern engineering practices such as telecommunications, information technology, and cybersecurity. The two fundamental properties of quantum mechanics: superposition (see Section 3) and entanglement (see Jansson [7], Section 4, where the theory of special relativity is reconstructed in a six-dimensional Euclidean space), receive not only a new interpretation but also a new mathematical framework. I hope this will lead to new insights in quantum information science.

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Additional information

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