Chapter

Perspective Chapter: Families of Seventh-Order KdV Equations Having Traveling Wave and Soliton Solutions

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Abstract

In this paper, we consider the problem of finding traveling wave solutions to the generalized seventh-order KdV equation (KdV7). Solitons are non-linear waves that exhibit extremely unexpected and interesting behavior—solitary waves that propagate without deformation. We use different approaches in order to find one and multisoliton solutions. Soliton travels through liquid, solid, and gaseous media and even as electron waves through an electromagnetic field. Making use of a traveling wave transformation, we obtain a non-linear ode, which is solved using either hyperbolic or elliptic algorithm. We also use the Hirota method to get the bilinear form, and then we may obtain multisoliton solutions. In the end, we consider the forced KdV7.

Keywords: traveling wave solutions, KdV7, solitons, cnoidal waves, deformed sine-Gordon equation, Sawada-Kotera equation, Kaup–Kupershmidt equation, Ito equation, lLax equation

1. Introduction

One of the most notable achievements in the second half of the twentieth century, which also clearly illustrates the underlying unity of Mathematics and Nonlinear Physics, is the Theory of Solitons. Solitons are nonlinear waves exhibiting extremely unexpected and interesting behavior—solitary waves propagating without deformation.

The other waves, the nonlinear ones, are less familiar and are very different from the linear ones. A wave in the sea approaching the shore is a good example of a nonlinear wave. Note that the amplitude, wavelength, and speed vary as the wave advances, while in linear waves, these are constant. The distance between the crests decreases, the height of the waves increases as they perceive the bottom, and the speed changes. The upper part of the wave overtakes the lower part, falls on it, and the wave breaks. There are even more intricate phenomena such as two waves that intersect, interact in complicated and nonlinear ways, and give rise to three waves instead of two.

Now we come to solitons. During a horseback ride around Edinburgh, on the Union Canal in Hermiston, very close to the Riccarton campus of Heriot-Watt University, the Scottish engineer John Scott-Russell watched as a barge was towed along a narrow canal by two horses that pulled from land to obtain a more efficient design of boats.

A decisive step in the theory of integrable systems was the integration of the KdV equation. Thus, Gardner, Greene, Kruskal, and Miura observed that if we consider a potential $u(x)$ for the stationary Schrödinger equation on the line, the corresponding scattering data are transformed extremely easily when the potential changes as long as $u(x, t)$ satisfies the KdV equation. Therefore, given an initial condition $u(x)$ for KdV, we can find the associated scattering data and determine its evolution immediately.

In this paper, we consider the following generalized seventh-order KdV equation (KdV7 for short):

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2\, u_{3x} + \alpha u_{2x}u_{3x} + \beta u_xu_{4x} + \gamma uu_{5x} + u_{7x} = 0. \quad (1)$$

This nonlinear evolution equation describes the behavior of physical phenomena such as shallow water waves and plasmas. Its conservation laws were determined to predict its complete integrability [1, 2]. In Ref. [3], Wazwaz obtained one and two soliton solutions for the following special cases:

- The seventh-order Sawada-Kotera-Ito equation:

$$u_t + 252u^3u_x + 63u_x^3 + 378u_xu_{2x} + 126u^2\, u_{3x} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0. \quad (2)$$

- The seventh-order Lax equation:

$$u_t + 140u^3u_x + 70u_x^3 + 280u_xu_{2x} + 70u^2\, u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0. \quad (3)$$

- The seventh-order Kaup-Kuperschmidt equation

$$u_t + 2016u^3u_x + 630u_x^3 + 2268u_xu_{2x} + 504u^2\, u_{3x} + 252u_{2x}u_{3x} + 147u_xu_{4x} + 42uu_{5x} + u_{7x} = 0. \quad (4)$$

These three cases of the seventh-order KdV equation are completely integrable. This means that each of these equations admits an infinite number of conservation laws, and as a result, each gives rise to N-soliton solutions. We aim to describe the families of these KdV7 that admit soliton and cnoidal wave solutions.

2. Cnoidal wave solutions

Let

$$u(x,t) = p + q\wp(x - \lambda t + \xi_0; g_2, g_3). \quad (5)$$

DOI: http://dx.doi.org/10.5772/intechopen.1004789

Then

$$
\begin{aligned}
&u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2\,u_{3x} + \alpha u_{2x}u_{3x} + \beta u_xu_{4x} + \gamma uu_{5x} + u_{7x} = \\
&\frac{1}{2}q\sqrt{-g_2\wp - g_3 + 4\wp^3}[2ap^3 - 2bg_3q^2 - cg_2pq - 36\gamma g_2p - 24\beta g_3q - 1440g_3 - 2\lambda \\
&+ (6ap^2q - 2bg_2q^2 - cg_2q^2 + 24dp^2 - 12\alpha g_2q - 36\beta g_2q - 36\gamma g_2q - 4032g_2)\wp + \\
&6p\left(aq^2 + 2cq + 120\gamma + 8dq\right)\wp^2 + \\
&2\left(aq^3 + 4bq^2 + 6cq^2 + 12dq^2 + 72\alpha q + 120\beta q + 360\gamma q + 20160\right)\wp^3],
\end{aligned}
\tag{5a}
$$

where $\wp = \wp(x - \lambda t + \xi_0; g_2, g_3)$.
The system to be solved is

$$
\begin{aligned}
&2ap^3 - 2bg_3q^2 - cg_2pq - 36\gamma g_2 - 24\beta g_3q - 1440g_3 + 2\lambda = 0. \\
&6ap^2q - 2bg_2q^2 - cg_2q^2 + 24dp^2 - 12\alpha g_2q - 36\beta g_2q - 4032g_2 = 0. \\
&6apq^2 + 12cpq + 720\gamma + 48dpq = 0. \\
&2aq^3 + 8bq^2 + 12cq^2 + 24dq^2 + 144\alpha q + 240\beta q + 40320 = 0.
\end{aligned}
\tag{6}
$$

We will have a solution for the following parameter values:

$$
\begin{aligned}
&g_2 = \frac{6p^2(aq + 4d)}{2bq^2 + cq^2 + 12\alpha q + 36\beta q + 4032}. \\
&g_3 = \frac{2ap^3 - cg_2pq - 36\gamma g_2 + 2\lambda}{2\left(bq^2 + 12\beta q + 720\right)}. \\
&p = -\frac{120\gamma}{q(aq + 2c + 8d)}. \\
&aq^3 + 2(2b + 3c + 6d)q^2 + 24\,(3\alpha + 5\beta)\,q + 20160 = 0.
\end{aligned}
\tag{7}
$$

Example. Let

$$
\begin{aligned}
&u^{(0,1)}(x,t) - 63.5567u(x,t)^3u^{(1,0)}(x,t) - 5089.7u^{(1,0)}(x,t)^3 + 0.939611u(x,t)u^{(1,0)}(x,t)u^{(2,0)}(x,t) \\
&+0.471886u\,(x,t)^2u^{(3,0)}(x,t) + 0.0310296u^{(2,0)}(x,t)u^{(3,0)}(x,t) + 0.626069u^{(1,0)}(x,t)u^{(4,0)}(x,t) \\
&+0.48252u(x,t)u^{(5,0)}(x,t) + u^{(7,0)}(x,t) = 0.
\end{aligned}
\tag{8}
$$

See **Figure 1**.
Let now

$$
u(x,t) = B + C\,\mathrm{cn}^2(\omega x - \lambda t + \xi_0, m). \tag{9}
$$

Then

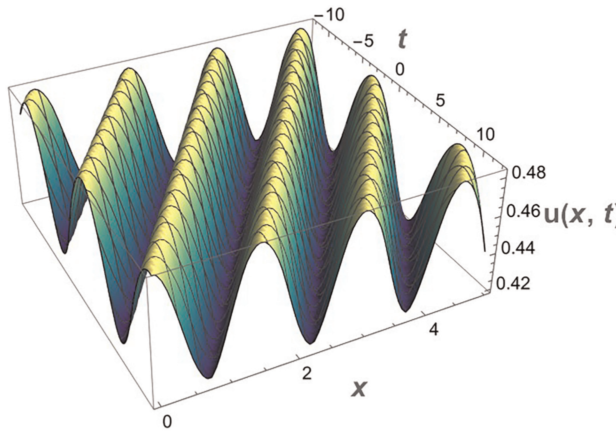

Figure 1.
$u(x,t) = 0.480062 + \wp(-7.21261t - x - (4.70653 - 5.33465i); 0.0139651, -0.0000274235)$.

$$
u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2\,u_{3x} + \alpha u_{2x}u_{3x} + \beta u_xu_{4x} + \gamma uu_{5x} + u_{7x} =
$$

$$
\begin{aligned}
&2C\,\mathrm{cn}\,\mathrm{sn}\,\mathrm{dn}\,[-\begin{pmatrix} \lambda - aB^3\omega - 2BcC\omega^3 + 4B^2d\omega^3 + 2BcCm\omega^3 - 8B^2dm\omega^3 + 8C\alpha\omega^5 - 24Cm\alpha\omega^5 + 16Cm^2\alpha\omega^5 \\ +8C\beta\omega^5 - 24Cm\beta\omega^5 + 16Cm^2\beta\omega^5 - 16B\gamma\omega^5 + 136Bm\gamma\omega^5 - 136Bm^2\gamma\omega^5 + 64\omega^7 \\ -2112m\omega^7 + 5952m^2\omega^7 - 3968m^3\omega^7 \end{pmatrix}\mathrm{cn} \\
&+\omega\begin{pmatrix} 3aB^2C - 4BcC\omega^2 + 4bC^2\omega^2 + 2cC^2\omega^2 - 8BCd\omega^2 + 8BcCm\omega^2 - 4bC^2m\omega^2 - 2cC^2m\omega^2 - 12B^2dm\omega^2 \\ +16BCdm\omega^2 + 16C\alpha\omega^4 - 88Cm\alpha\omega^4 + 88Cm^2\alpha\omega^4 + 16C\beta\omega^4 - 136Cm\beta\omega^4 + 136Cm^2\beta\omega^4 + 16C\gamma\omega^4 \\ +240Bm\gamma\omega^4 - 136Cm\gamma\omega^4 - 480Bm^2\gamma\omega^4 + 136Cm^2\gamma\omega^4 - 4032m\omega^6 + 24192m^2\omega^6 - 24192m^3\omega^6 \end{pmatrix}\mathrm{cn}^3 \\
&+\omega\begin{pmatrix} 3aBC^2 - 4bC^2\omega^2 - 4cC^2\omega^2 - 4C^2d\omega^2 - 6BcCm\omega^2 + 8bC^2m\omega^2 + 8cC^2m\omega^2 - 24BCdm\omega^2 \\ +8C^2dm\omega^2 + 72Cm\alpha\omega^4 - 144Cm^2\alpha\omega^4 + 120Cm\beta\omega^4 - 240Cm^2\beta\omega^4 + 240Cm\gamma\omega^4 + 360Bm^2\gamma\omega^4 \\ -480Cm^2\gamma\omega^4 - 20160m^2\omega^6 + 40320m^3\omega^6 \end{pmatrix}\mathrm{cn}^5 \\
&+\omega\left(aC^3 - 4bC^2m\omega^2 - 6cC^2m\omega^2 - 12C^2dm\omega^2 + 72Cm^2\alpha\omega^4 + 120Cm^2\beta\omega^4 + 360Cm^2\gamma\omega^4 - 20160m^3\omega^6\right)\mathrm{cn}^7].
\end{aligned}
$$

Next, we equate to zero the coefficients of cn^j to obtain an algebraic system of nonlinear equations. This system admits a solution under the condition

$$
\Delta_1\Delta_2 = 0, \tag{10}
$$

where

$$
\begin{aligned}
\Delta_1 = {}& 17640a^2 + 27a\alpha^2\gamma + 90a\alpha\beta\gamma + 180a\alpha\gamma^2 + 75a\beta^2\gamma + 300a\beta\gamma^2 - 840ab\gamma + 300a\gamma^3 - 126a\alpha c - 210a\beta c \\
& -1260a\gamma c - 504a\alpha d - 840a\beta d - 2520a\gamma d + 10b^2\gamma^2 + 14bc^2 - 3\alpha b\gamma c - 5b\beta\gamma c + 10b\gamma^2c + 112bcd + 224bd^2 \\
& -12\alpha b\gamma d - 20b\beta\gamma d - 20b\gamma^2d + 14c^3 - 3\alpha\gamma c^2 - 5\beta\gamma c^2 + 126c^2d + 336cd^2 - 15\alpha\gamma cd - 25\beta\gamma cd - 30\gamma^2cd + 224d^3 \\
& -12\alpha\gamma d^2 - 20\beta\gamma d^2 - 30\gamma^2d^2, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_2 = {}& 28224a^2 + 3a\alpha^3 + 3a\alpha^2\beta + 63a\alpha^2\gamma - 63a\alpha\beta^2 + 234a\alpha\beta\gamma + 297a\alpha\gamma^2 - 135a\beta^3 + 135a\beta^2\gamma + 168a\alpha b \\
& +3192a\beta b + 675a\beta\gamma^2 - 3528ab\gamma + 405a\gamma^3 - 252a\alpha c + 588a\beta c - 2772ac\gamma - 1008a\alpha d - 3024a\beta d - 3024a\gamma d \\
& +504b^3 + 4\alpha^2b^2 - 16\alpha\beta b^2 + 84\alpha b^2\gamma - 20\beta^2b^2 + -60\beta b^2\gamma + 360b^2\gamma^2 + 84b^2c - 2016b^2d - 70bc^2 + 2\alpha^2bc \\
& -12\alpha\beta bc + 48\alpha bc\gamma + 10\beta^2bc - 120\beta bc\gamma + 270bc\gamma^2 - 672bcd + 2016bd^2 - 6\alpha^2bd + 12\alpha\beta bd - 108\alpha b\gamma d \\
& +90\beta^2bd - 180\beta b\gamma d - 270b\gamma^2d + 7c^3 - 2\alpha\beta c^2 + 3\alpha c^2\gamma + 10\beta^2c^2 - 45\beta c^2\gamma + 45c^2\gamma^2 + 168c^2d + 1008cd^2 \\
& -3\alpha^2cd + 6\alpha\beta cd - 54\alpha c\gamma d + 45\beta^2cd - 90\beta c\gamma d - 135c\gamma^2d.
\end{aligned}
$$

The Eqs. (2)–(4) obey the condition in Eq. (10).

Solving the system we obtain the solutions as follows:

$$\lambda = \omega \begin{pmatrix} aB^3 + 8B^2dm\omega^2 - 4B^2d\omega^2 - 2BcCm\omega^2 + 2BcC\omega^2 \\ +16B\gamma\omega^4 + 136B\gamma m^2\omega^4 - 136B\gamma m\omega^4 - 8\alpha C\omega^4 \\ -8\beta C\omega^4 - 16\alpha Cm^2\omega^4 - 16\beta Cm^2\omega^4 + 24\alpha Cm\omega^4 \\ +24\beta Cm\omega^4 + 3968m^3\omega^6 - 5952m^2\omega^6 + 2112m\omega^6 - 64\omega^6 \end{pmatrix}.$$

$$B = -\frac{4(2m-1)\omega^2}{3\left(aC^2 - 2cCm\omega^2 - 8Cdm\omega^2 + 120\gamma m^2\omega^4\right)}\left(bC^2 + cC^2 + C^2d - 18\alpha Cm\omega^2 - 30\beta Cm\omega^2\right.$$
$$\left. -60\gamma Cm\omega^2 + 5040m^2\omega^4\right). \tag{11}$$

$$aC^3 + \left(-4bm\omega^2 - 6cm\omega^2 - 12dm\omega^2\right)C^2 + \left(72\alpha m^2\omega^4 + 120\beta m^2\omega^4 + 360\gamma m^2\omega^4\right)C$$
$$-20160m^3\omega^6 = 0.$$

- Sawada-Kotera-Ito Eq. (2):

$$C = 2m\omega^2$$
$$\lambda = 4\omega\left(63B^3 + 252B^2m\omega^2 - 126B^2\omega^2 + 336Bm^2\omega^4 - 336Bm\omega^4 + 84B\omega^4\right.$$
$$\left. +152m^3\omega^6 - 228m^2\omega^6 + 108m\omega^6 - 16\omega^6\right). \tag{12}$$
$$u(x,t) = B + C\mathrm{cn}^2\left(\sqrt{\omega}x - \lambda t\right) + \xi_0|m).$$

$$B = -\frac{4}{3}(2m-1)\omega^2, C = 4m\omega^2.$$
$$\lambda = \frac{128}{3}(m-2)(m+1)(2m-1)\omega^7. \tag{13}$$
$$u(x,t) = B + C\mathrm{cn}^2\left(\sqrt{\omega}x - \lambda t\right) + \xi_0|m).$$

- Lax Eq. (3):

$$C = 2m\omega^2.$$
$$\lambda = 4\omega\left(35B^3 + 140B^2m\omega^2 - 70B^2\omega^2 + 196Bm^2\omega^4 - 196Bm\omega^4 + 56B\omega^4 + 96m^3\omega^6\right. \tag{14}$$
$$\left. -144m^2\omega^6 + 80m\omega^6 - 16\omega^6\right)u(x,t) = B + 2m\omega\mathrm{cn}^2\left(\sqrt{\omega}(x - \lambda t) + \xi_0|m\right).$$

- Kaup-Kuperschmidt Eq. (4):

$$B = -\frac{1}{6}(2m-1)\omega^2, C = \frac{m\omega^2}{2}.$$
$$\lambda = \frac{2}{3}(m-2)(m+1)(2m-1)\omega^7. \tag{15}$$
$$u(x,t) = B + 2m\omega\mathrm{cn}^2\left(\sqrt{\omega}(x - \lambda t) + \xi_0|m\right).$$

- Letting $m = 1$ in Eqs. (2)–(4), we obtain solitonic solutions.

3. Soliton solutions

3.1 First family

Let

$$A = \frac{252}{\alpha+\beta+\gamma}, w = k^7.$$
$$c = -\frac{1}{126}(\alpha+\beta+\gamma)(\alpha-5(\beta+4\gamma)) - b - d. \tag{16}$$
$$d = \frac{42a}{\alpha+\beta+\gamma} + \frac{1}{378}(\alpha+\beta+\gamma)(\alpha-5\beta+10\gamma) + \frac{b}{3}.$$

We make the transformation

$$u = A\partial_{x,x}\log(1+\exp(kx-wt)) \tag{17}$$

to obtain the soliton solution

$$u_{\text{soliton}}(x,t) = \frac{126k^2}{(\alpha+\beta+\gamma)\left(1+\cosh\left(kx-k^7t-\xi\right)\right)}. \tag{18}$$

For a graphical illustration, see **Figure 2**.
We are interested in the existence of two soliton solutions. Let

$$u(x,t) = \frac{252}{\alpha+\beta+\gamma}\partial_{x,x}\log(1+\exp\theta_1+\exp\theta_2+\rho\exp(\theta_1+\theta_2)).$$
$$\theta_1 = k_1x - k_1^7t \text{ and } \theta_2 = k_2x - k_2^7t \tag{19}$$

Making the choices

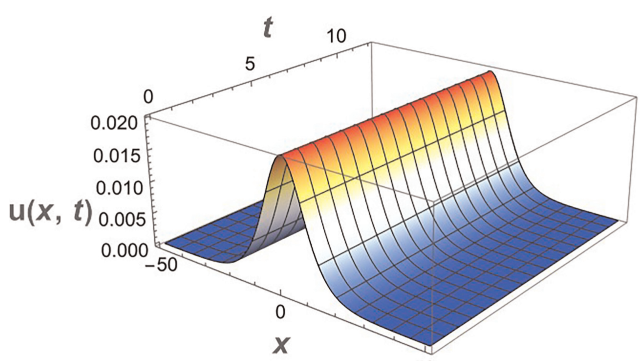

Figure 2.
Soliton solution $u(x,t) = \frac{0.04}{1+\cosh(0.0000128t-0.2x)}$ *to the Sawada-Kotera-Ito equation with* k = 0.2.

$$a = -\frac{\gamma^2(\beta+2\gamma)\left(\beta^3-3\beta^2\gamma-9\beta\gamma^2+31\gamma^3\right)}{49(\beta-5\gamma)^3}$$

$$b = -\frac{(\beta-\gamma)\left(\beta^3-6\beta^2\gamma+15\beta\gamma^2-23\gamma^3\right)}{7(\beta-5\gamma)^2}$$

$$c = \frac{\beta^4-9\beta^3\gamma+25\beta^2\gamma^2-33\beta\gamma^3+76\gamma^4}{7(\beta-5\gamma)^2}$$

$$d = \frac{2\gamma\left(\beta^3-5\beta^2\gamma+2\beta\gamma^2+17\gamma^3\right)}{7(\beta-5\gamma)^2}$$

$$\alpha = -\frac{\beta^2-4\beta\gamma+13\gamma^2}{\beta-5\gamma}$$

$$\rho = \frac{\gamma^2(k_1-k_2)^2\left(k_1{}^2-k_1k_2+k_2{}^2\right)^2}{(k_1+extk_2)^2\left(2\beta^2k_1{}^2k_2{}^2-2\beta\gamma k_1k_2\left(k_1{}^2+4k_1k_2+k_2{}^2\right)+\gamma^2\left(k_1{}^4+4k_1{}^3k_2+9k_1{}^2k_2{}^2+4k_1k_2{}^3+k_2{}^4\right)\right)}. \tag{20}$$

We obtain

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2\,u_{3x} + \alpha\,u_{2x}u_{3x} + \beta u_xu_{4x} + \gamma u_{5x} + u_{7x} = (\beta-\gamma)(\beta-2\gamma)(\beta-3\gamma)R(k_1,k_2,\theta_1,\theta_2).$$

We conclude that the two soliton solutions exist for the parameter values in Eq. (20) under the condition

$$(\beta-\gamma)(\beta-2\gamma)(\beta-3\gamma) = 0. \tag{21}$$

We obtained the following result:

Theorem. The following families of KdV7 admit one and two soliton solutions for any p. The two soliton solutions have the form

$$u_{2-\text{soliton}}(x,t) = \frac{252}{\alpha+\beta+\gamma}\partial_{xx}\log\left(1+\exp(\theta_1)+\exp(\theta_2)+\rho\exp(\theta_1+\theta_2)\right), \tag{22}$$

being

$$\theta_1 = k_1x - k_1^7t, \theta_2 = k_2x - k_2^7t. \tag{23}$$

- First set:

$$\left\{\begin{array}{c} a = \frac{15p^3}{784}, b = 0, c = \frac{15p^2}{28}, d = \frac{15p^2}{56}, \alpha = \frac{5p}{2}, \beta = p, \\ \rho = \frac{(k_1-k_2)^2\left(k_1{}^2-k_1k_2+k_2{}^2\right)^2}{(k_1+k_2)^2\left(k_1{}^2+k_1k_2+k_2{}^2\right)^2}, \gamma = p \end{array}\right\} \tag{24}$$

KdV7:

$$u_t + \frac{15}{784}p^3u^3u_x + \frac{15}{28}p^2uu_xu_{2x} + \frac{15}{56}p^2u^2u_{3x} + \frac{5}{2}pu_{2x}u_{3x} + pu_xu_{4x} + puu_{5x} + u_{7x} = 0 \tag{25}$$

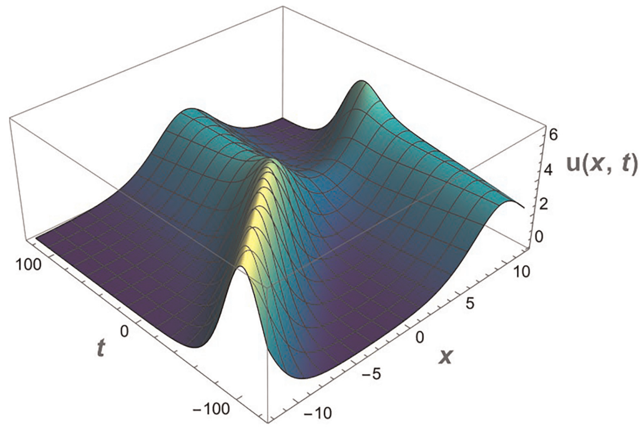

Figure 3.
Two soliton solution for p = *1*.

An illustration with $p = 1$ is shown in **Figure 3**. The solution is

$$u(x,t) = \frac{14.e^{0.352855t+0.5x} + 27.44e^{0.278313t+0.7x} + 2.52676e^{0.2705t+1.2x} + 0.0975793e^{0.262688t+1.7x} + 0.0497854e^{0.188146t+1.9x}}{\left(e^{0.172521t+0.5x} + e^{0.0979793t+0.7x} + 0.0035561e^{0.0901668t+1.2x} + e^{0.180334t}\right)^2} \tag{26}$$

- Second set:

$$\left\{ \begin{array}{c} a = \frac{4p^3}{147}, b = \frac{p^2}{7}, c = \frac{6p^2}{7}, d = \frac{2p^2}{7}, \alpha = 3p, \beta = 2p, \\ \rho = \frac{(k_1 - k_2)^2 \left({k_1}^2 - k_1 k_2 + {k_2}^2\right)}{(k_1 + k_2)^2 \left({k_1}^2 + k_1 k_2 + {k_2}^2\right)}, \gamma = p \end{array} \right\} \tag{27}$$

KdV7:

$$u_t + \frac{4}{147} p^3 u^3 u_x + \frac{1}{7} p^2 u_x^3 + \frac{6}{7} p^2 u u_x u_{2x} + \frac{2}{7} p^2 u^2 u_{3x} + 3p u_{2x} u_{3x} + 2p u_x u_{4x} + p u u_{5x} + u_{7x} = 0 \tag{28}$$

- Third set:

$$\left\{ \begin{array}{c} a = \frac{5p^3}{98}, b = \frac{5p^2}{14}, c = \frac{10p^2}{7}, d = \frac{5p^2}{14}, \alpha = 5p, \beta = 3p, \\ \rho = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \gamma = p \end{array} \right\} \tag{29}$$

KdV7:

$$u_t + \frac{5}{98} p^3 u^3 u_x + \frac{5}{14} p^2 u_x^3 + \frac{10}{7} p^2 u u_x u_{2x} + \frac{5}{14} p^2 u^2 u_{3x} + 5p u_{2x} u_{3x} + 3p u_x u_{4x} + p u u_{5x} + u_{7x} = 0 \tag{30}$$

Now, our aim is to find three soliton solutions for the parameter values in Eq. (20). Assume the ansatz

$$u(x,t) = \frac{252}{\alpha+\beta+\gamma}\partial_{xx}\log\left(1+\sum_{j=1}^{3}\eta_j + \sum_{1\le i<j\le 3}\rho_{i,j}\eta_i\eta_j + \rho_{1,2,3}\eta_1\eta_2\eta_3\right), \tag{31}$$

where

$$\eta_j = \exp\left(k_j x - k_j^7 t\right) \text{ for } j = 1,2,3. \tag{32}$$

We have:

$$\begin{aligned}
&u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2\,u_{3x} + \alpha\,u_{2x}u_{3x} + \beta u_xu_{4x} + \gamma uu_{5x} + u_{7x} = \\
&98(\beta-5\gamma)k_1k_2(k_1+k_2)\,(\gamma^2k_1^6 - \gamma^2\rho_{1,2}k_1^6 - 6\gamma^2k_2k_1^5 + 2\beta\gamma k_2k_1^5 - 4\gamma^2k_2\rho_{1,2}k_1^5 \\
&+2\beta^2k_2^2k_1^4 + 18\gamma^2k_2^2k_1^4 - 12\beta\gamma k_2^2k_1^4 - 8\gamma^2k_2^2\rho_{1,2}k_1^4 - 4\beta^2k_2^3k_1^3 - 26\gamma^2k_2^3k_1^3 + 20\beta\gamma k_2^3k_1^3 \\
&-10\gamma^2k_2^3\rho_{1,2}k_1^3 + 2\beta^2k_2^4k_1^2 + 18\gamma^2k_2^4k_1^2 - 12\beta\gamma k_2^4k_1^2 - 8\gamma^2k_2^4\rho_{1,2}k_1^2 - 6\gamma^2k_2^5k_1 + 2\beta\gamma k_2^5k_1 \\
&-4\gamma^2k_2^5\rho_{1,2}k_1 + \gamma^2k_2^6 - \gamma^2k_2^6\rho_{1,2})/\gamma^4z_1z_2 \\
&+98(\beta-5\gamma)k_1k_3(k_1+k_3)\,(\gamma^2k_1^6 - \gamma^2\rho_{1,3}k_1^6 - 6\gamma^2k_3k_1^5 + 2\beta\gamma k_3k_1^5 - 4\gamma^2k_3\rho_{1,3}k_1^5 \\
&+2\beta^2k_3^2k_1^4 + 18\gamma^2k_3^2k_1^4 - 12\beta\gamma k_3^2k_1^4 - 8\gamma^2k_3^2\rho_{1,3}k_1^4 - 4\beta^2k_3^3k_1^3 - 26\gamma^2k_3^3k_1^3 + 20\beta\gamma k_3^3k_1^3 - \\
&10\gamma^2k_3^3\rho_{1,3}k_1^3 + 2\beta^2k_3^4k_1^2 + 18\gamma^2k_3^4k_1^2 - 12\beta\gamma k_3^4k_1^2 - 8\gamma^2k_3^4\rho_{1,3}k_1^2 - 6\gamma^2k_3^5k_1 + 2\beta\gamma k_3^5k_1 \\
&-4\gamma^2k_3^5\rho_{1,3}k_1 + \gamma^2k_3^6 - \gamma^2k_3^6\rho_{1,3})/\gamma^4z_1z_3 \\
&+98(\beta-5\gamma)k_2k_3(k_2+k_3)\,(\gamma^2k_2^6 - \gamma^2\rho_{2,3}k_2^6 - 6\gamma^2k_3k_2^5 + 2\beta\gamma k_3k_2^5 - 4\gamma^2k_3\rho_{2,3}k_2^5 \\
&+2\beta^2k_3^2k_2^4 + 18\gamma^2k_3^2k_2^4 - 12\beta\gamma k_3^2k_2^4 - 8\gamma^2k_3^2\rho_{2,3}k_2^4 - 4\beta^2k_3^3k_2^3 - 26\gamma^2k_3^3k_2^3 + 20\beta\gamma k_3^3k_2^3 \\
&-10\gamma^2k_3^3\rho_{2,3}k_2^3 + 2\beta^2k_3^4k_2^2 + 18\gamma^2k_3^4k_2^2 - 12\beta\gamma k_3^4k_2^2 - 8\gamma^2k_3^4\rho_{2,3}k_2^2 - 6\gamma^2k_3^5k_2 + 2\beta\gamma k_3^5k_2 \\
&-4\gamma^2k_3^5\rho_{2,3}k_2 + \gamma^2k_3^6 - \gamma^2k_3^6\rho_{2,3})/\gamma^4z_2z_3 + \text{h.o.t}
\end{aligned}$$

Equating to zero the coefficients of z_1z_2, z_1z_3, and z_2z_3 and solving the resulting system of algebraic equations we obtain

$$\begin{aligned}
\rho_{1,2} &= \frac{(k_1-k_2)^2\left(2\beta^2k_2^2k_1^2 + 2\beta\gamma k_2k_1^3 - 8\beta\gamma k_2^2k_1^2 + 2\beta\gamma k_2^3k_1 + \gamma^2k_1^4 - 4\gamma^2k_2k_1^3 + 9\gamma^2k_2^2k_1^2 - 4\gamma^2k_2^3k_1 + \gamma^2k_2^4\right)}{\gamma^2(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)^2}. \\
\rho_{1,3} &= \frac{(k_1-k_3)^2\left(2\beta^2k_3^2k_1^2 + 2\beta\gamma k_3k_1^3 - 8\beta\gamma k_3^2k_1^2 + 2\beta\gamma k_3^3k_1 + \gamma^2k_1^4 - 4\gamma^2k_3k_1^3 + 9\gamma^2k_3^2k_1^2 - 4\gamma^2k_3^3k_1 + \gamma^2k_3^4\right)}{\gamma^2(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)^2}. \\
\rho_{2,3} &= \frac{(k_2-k_3)^2\left(2\beta^2k_3^2k_2^2 + 2\beta\gamma k_3k_2^3 - 8\beta\gamma k_3^2k_2^2 + 2\beta\gamma k_3^3k_2 + \gamma^2k_2^4 - 4\gamma^2k_3k_2^3 + 9\gamma^2k_3^2k_2^2 - 4\gamma^2k_3^3k_2 + \gamma^2k_3^4\right)}{\gamma^2(k_2+k_3)^2\left(k_2^2+k_3k_2+k_3^2\right)^2}.
\end{aligned} \tag{33}$$

Next, we equate to zero the coefficient of $z_1z_2z_3$ in order to obtain the value for $\rho_{1,2,3}$. It is given as follows.

- For $\beta = \gamma$:

$$
\begin{aligned}
&\rho_{1,2,3} = P_1/Q_1, \text{where}\\
&P_1 = \gamma^4 (k_2 - k_1)^2 (k_2 - k_3)^2 (k_3 - k_1)^2 (k_1 + k_2 + k_3)\\
&(k_2^4 k_1^{13} + k_3^4 k_1^{13} - 2k_2 k_3^3 k_1^{13} + 3k_2^2 k_3^2 k_1^{13} - 2k_2^3 k_3 k_1^{13} + k_2^5 k_1^{12} + k_3^5 k_1^{12} - k_2 k_3^4 k_1^{12} + k_2^2 k_3^3 k_1^{12}\\
&+k_2^3 k_3^2 k_1^{12} - k_2^4 k_3 k_1^{12} + 2k_2^6 k_1^{11} + 2k_3^6 k_1^{11} - 3k_2 k_3^5 k_1^{11} + 6k_2^2 k_3^4 k_1^{11} - 5k_2^3 k_3^3 k_1^{11} + 6k_2^4 k_3^2 k_1^{11}\\
&-3k_2^5 k_3 k_1^{11} + 2k_2^7 k_1^{10} + 2k_3^7 k_1^{10} - 4k_2 k_3^6 k_1^{10} - 10k_2^2 k_3^5 k_1^{10} - 18k_2^3 k_3^4 k_1^{10} - 18k_2^4 k_3^3 k_1^{10} - 10k_2^5 k_3^2 k_1^{10}\\
&-4k_2^6 k_3 k_1^{10} + 3k_2^8 k_1^9 + 3k_3^8 k_1^9 - 6k_2 k_3^7 k_1^9 - 7k_2^2 k_3^6 k_1^9 - 38k_2^3 k_3^5 k_1^9 - 26k_2^4 k_3^4 k_1^9 - 38k_2^5 k_3^3 k_1^9 - 7k_2^6 k_3^2 k_1^9\\
&-6k_2^7 k_3 k_1^9 + 3k_2^9 k_1^8 + 3k_3^9 k_1^8 - 3k_2 k_3^8 k_1^8 - 7k_2^2 k_3^7 k_1^8 - 35k_2^3 k_3^6 k_1^8 - 58k_2^4 k_3^5 k_1^8 - 58k_2^5 k_3^4 k_1^8 - 35k_2^6 k_3^3 k_1^8\\
&-7k_2^7 k_3^2 k_1^8 - 3k_2^8 k_3 k_1^8 + 2k_2^{10} k_1^7 + 2k_3^{10} k_1^7 - 6k_2 k_3^9 k_1^7 - 7k_2^2 k_3^8 k_1^7 - 41k_2^3 k_3^7 k_1^7 - 60k_2^4 k_3^6 k_1^7 - 99k_2^5 k_3^5 k_1^7\\
&-60k_2^6 k_3^4 k_1^7 - 41k_2^7 k_3^3 k_1^7 - 7k_2^8 k_3^2 k_1^7 - 6k_2^9 k_3 k_1^7 + 2k_2^{11} k_1^6 + 2k_3^{11} k_1^6 - 4k_2 k_3^{10} k_1^6 - 7k_2^2 k_3^9 k_1^6 - 35k_2^3 k_3^8 k_1^6\\
&-60k_2^4 k_3^7 k_1^6 - 92k_2^5 k_3^6 k_1^6 - 92k_2^6 k_3^5 k_1^6 - 60k_2^7 k_3^4 k_1^6 - 35k_2^8 k_3^3 k_1^6 - 7k_2^9 k_3^2 k_1^6 - 4k_2^{10} k_3 k_1^6 + k_2^{12} k_1^5 + k_3^{12} k_1^5\\
&-3k_2 k_3^{11} k_1^5 - 10k_2^2 k_3^{10} k_1^5 - 38k_2^3 k_3^9 k_1^5 - 58k_2^4 k_3^8 k_1^5 - 99k_2^5 k_3^7 k_1^5 - 92k_2^6 k_3^6 k_1^5 - 99k_2^7 k_3^5 k_1^5 - 58k_2^8 k_3^4 k_1^5\\
&-38k_2^9 k_3^3 k_1^5 - 10k_2^{10} k_3^2 k_1^5 - 3k_2^{11} k_3 k_1^5 + k_2^{13} k_1^4 + k_3^{13} k_1^4 - k_2 k_3^{12} k_1^4 + 6k_2^2 k_3^{11} k_1^4 - 18k_2^3 k_3^{10} k_1^4 - 26k_2^4 k_3^9 k_1^4\\
&-58k_2^5 k_3^8 k_1^4 - 60k_2^6 k_3^7 k_1^4 - 60k_2^7 k_3^6 k_1^4 - 58k_2^8 k_3^5 k_1^4 - 26k_2^9 k_3^4 k_1^4 - 18k_2^{10} k_3^3 k_1^4 + 6k_2^{11} k_3^2 k_1^4 - k_2^{12} k_3 k_1^4\\
&-2k_2 k_3^{13} k_1^3 + k_2^2 k_3^{12} k_1^3 - 5k_2^3 k_3^{11} k_1^3 - 18k_2^4 k_3^{10} k_1^3 - 38k_2^5 k_3^9 k_1^3 - 35k_2^6 k_3^8 k_1^3 - 41k_2^7 k_3^7 k_1^3 - 35k_2^8 k_3^6 k_1^3\\
&-38k_2^9 k_3^5 k_1^3 - 18k_2^{10} k_3^4 k_1^3 - 5k_2^{11} k_3^3 k_1^3 + k_2^{12} k_3^2 k_1^3 - 2k_2^{13} k_3 k_1^3 + 3k_2^2 k_3^{13} k_1^2 + k_2^3 k_3^{12} k_1^2 + 6k_2^4 k_3^{11} k_1^2\\
&-10k_2^5 k_3^{10} k_1^2 - 7k_2^6 k_3^9 k_1^2 - 7k_2^7 k_3^8 k_1^2 - 7k_2^8 k_3^7 k_1^2 - 7k_2^9 k_3^6 k_1^2 - 10k_2^{10} k_3^5 k_1^2 + 6k_2^{11} k_3^4 k_1^2 + k_2^{12} k_3^3 k_1^2\\
&+3k_2^{13} k_3^2 k_1^2 - 2k_2^3 k_3^{13} k_1 - k_2^4 k_3^{12} k_1 - 3k_2^5 k_3^{11} k_1 - 4k_2^6 k_3^{10} k_1 - 6k_2^7 k_3^9 k_1 - 3k_2^8 k_3^8 k_1 - 6k_2^9 k_3^7 k_1 - 4k_2^{10} k_3^6 k_1\\
&-3k_2^{11} k_3^5 k_1 - k_2^{12} k_3^4 k_1 - 2k_2^{13} k_3^3 k_1 + k_2^4 k_3^{13} + k_2^5 k_3^{12} + 2k_2^6 k_3^{11} + 2k_2^7 k_3^{10} + 3k_2^8 k_3^9 + 3k_2^9 k_3^8 + 2k_2^{10} k_3^7\\
&+2k_2^{11} k_3^6 + k_2^{12} k_3^5 + k_2^{13} k_3^4),
\end{aligned}
$$

and

$$
\begin{aligned}
&Q_1 = \gamma^4 (k_1 + k_2)^2 \left(k_1^2 + k_2 k_1 + k_2^2\right)^2 (k_1 + k_3)^2 (k_2 + k_3)^2 (k_1 + k_2 + k_3)^2 \left(k_1^2 + k_3 k_1 + k_3^2\right)^2\\
&\left(k_2^2 + k_3 k_2 + k_3^2\right)^2\\
&(k_1^4 + 2k_2 k_1^3 + 2k_3 k_1^3 + 3k_2^2 k_1^2 + 3k_3^2 k_1^2 + 5k_2 k_3 k_1^2 + 2k_2^3 k_1 + 2k_3^3 k_1 + 5k_2 k_3^2 k_1 + 5k_2^2 k_3 k_1\\
&+k_2^4 + k_3^4 + 2k_2 k_3^3 + 3k_2^2 k_3^2 + 2k_2^3 k_3).
\end{aligned}
$$

- For $\beta = 2\gamma$:

$$
\begin{aligned}
&\rho_{1,2,3} = P_2/Q_2, \text{where}\\
&P_2 = \gamma^4 (k_2 - k_1)^2 \left(k_1^2 - k_2 k_1 + k_2^2\right) \left(k_1^2 + k_2 k_1 + k_2^2\right) (k_2 - k_3)^2 (k_3 - k_1)^2 (k_1 + k_2 + k_3)^2\\
&\left(k_1^2 - k_3 k_1 + k_3^2\right) \left(k_1^2 + k_3 k_1 + k_3^2\right) \left(k_2^2 - k_3 k_2 + k_3^2\right) \left(k_2^2 + k_3 k_2 + k_3^2\right) (k_1^4 + 2k_2 k_1^3 + 2k_3 k_1^3\\
&+3k_2^2 k_1^2 + 3k_3^2 k_1^2 + 5k_2 k_3 k_1^2 + 2k_2^3 k_1 + 2k_3^3 k_1 + 5k_2 k_3^2 k_1 + 5k_2^2 k_3 k_1 + k_2^4 + k_3^4 + 2k_2 k_3^3 + 3k_2^2 k_3^2 + 2k_2^3 k_3),
\end{aligned}
$$

and

$$
\begin{aligned}
&Q_2 = \gamma^4 (k_1 + k_2)^2 \left(k_1^2 + k_2 k_1 + k_2^2\right)^2 (k_1 + k_3)^2 (k_2 + k_3)^2 (k_1 + k_2 + k_3)^2 \left(k_1^2 + k_3 k_1 + k_3^2\right)^2\\
&\left(k_2^2 + k_3 k_2 + k_3^2\right)^2\\
&(k_1^4 + 2k_2 k_1^3 + 2k_3 k_1^3 + 3k_2^2 k_1^2 + 3k_3^2 k_1^2 + 5k_2 k_3 k_1^2 + 2k_2^3 k_1 + 2k_3^3 k_1 + 5k_2 k_3^2 k_1 + 5k_2^2 k_3 k_1 + k_2^4 + k_3^4\\
&+2k_2 k_3^3 + 3k_2^2 k_3^2 + 2k_2^3 k_3).
\end{aligned}
$$

- For $\beta = 3\gamma$:

$\rho_{1,2,3} = P_3/Q_3$, where

$$P_3 = \gamma^4 (k_2 - k_1)^2 \left(k_1^2 - k_2 k_1 + k_2^2\right)\left(k_1^2 + k_2 k_1 + k_2^2\right)(k_2 - k_3)^2 (k_3 - k_1)^2 (k_1 + k_2 + k_3)^2 \left(k_1^2 - k_3 k_1 + k_3^2\right)$$
$$\left(k_1^2 + k_3 k_1 + k_3^2\right)$$
$$\left(k_2^2 - k_3 k_2 + k_3^2\right)\left(k_2^2 + k_3 k_2 + k_3^2\right)(k_1^4 + 2k_2 k_1^3 + 2k_3 k_1^3 + 3k_2^2 k_1^2 + 3k_3^2 k_1^2 + 5k_2 k_3 k_1^2 + 2k_2^3 k_1 + 2k_3^3 k_1 + 5k_2 k_3^2 k_1$$
$$+5k_2^2 k_3 k_1 + k_2^4 + k_3^4 + 2k_2 k_3^3 + 3k_2^2 k_3^2 + 2k_2^3 k_3),$$

and

$$Q_3 = \gamma^4 (k_1 + k_2)^2 \left(k_1^2 + k_2 k_1 + k_2^2\right)^2 (k_1 + k_3)^2 (k_2 + k_3)^2 (k_1 + k_2 + k_3)^2 \left(k_1^2 + k_3 k_1 + k_3^2\right)^2 \left(k_2^2 + k_3 k_2 + k_3^2\right)^2$$
$$(k_1^4 + 2k_2 k_1^3 + 2k_3 k_1^3 + 3k_2^2 k_1^2 + 3k_3^2 k_1^2 + 5k_2 k_3 k_1^2 + 2k_2^3 k_1 + 2k_3^3 k_1 + 5k_2 k_3^2 k_1 + 5k_2^2 k_3 k_1 + k_2^4 + k_3^4$$
$$+2k_2 k_3^3 + 3k_2^2 k_3^2 + 2k_2^3 k_3).$$

We have three soliton solutions only when $\beta = 2\gamma$ or $\beta = 3\gamma$. Thus, the KdV7 has two soliton solutions for the parameter values in Eq. (20), but it does not have three soliton solutions for $\gamma = \beta$..

3.2 Second family

Let

$$\begin{aligned} a &= -\frac{5(7\alpha + 5\beta - 6\gamma)(\alpha + \beta + \gamma)^2}{7938}. \\ b &= \frac{1}{63}(\alpha + \beta + \gamma)(36\alpha + 35\beta + 10\gamma). \\ c &= -\frac{1}{63}(\alpha + \beta + \gamma)(37\alpha + 35\beta + 15\gamma). \\ d &= \frac{1}{126}(\alpha + \beta + \gamma)(\alpha + 5\beta + 30\gamma). \end{aligned} \tag{34}$$

We make the transformation

$$u(x,t) = \frac{126}{\alpha + \beta + \gamma}\partial_{x,x} \log\left(1 + \exp\left(kx - k^7 t\right) + \rho \exp\left(2kx - 2k^7 t\right)\right) \tag{35}$$

to obtain the soliton solution

$$u_{\text{soliton}}(x,t) = \frac{504k^2 e^{k^7 t + kx}}{(\alpha + \beta + \gamma)\left(2e^{k^7 t} + e^{kx}\right)^2} \text{for} \, \rho = \frac{1}{4}. \tag{36}$$

A cnoidal wave solution is

$$u_{\text{cnoidal}}(x,t) = p + \frac{\left(1 - 2m \pm \sqrt{m^2 - m + 1}\right)p}{m - 1} \text{cn}^2\left(\sqrt{\frac{q(\alpha + \beta + \gamma)}{252m}}(x - \lambda t) + \xi_0, m\right), \tag{37}$$

where

$$\lambda = -\frac{1}{500094m^3}[(\alpha+\beta+\gamma)^2(2205\alpha m^3p^3 + 1575\beta m^3p^3 - 1890\gamma m^3p^3 - 126\alpha m^3p^2q$$
$$-630\beta m^3p^2q - 3780\gamma m^3p^2q - 2331\alpha m^3pq^2 - 2205\beta m^3pq^2 - 2016\gamma m^3pq^2 + 2\alpha m^3q^3$$
$$+2\beta m^3q^3 - 124\gamma m^3q^3 + 63\alpha m^2p^2q + 315\beta m^2p^2q + 1890\gamma m^2p^2q + 2331\alpha m^2pq^2 + 2205\beta m^2pq^2$$
$$+2016\gamma m^2pq^2 - 3\alpha m^2q^3 - 3\beta m^2q^3 + 186\gamma m^2q^3 - 126\gamma mpq^2 - 3\alpha mq^3 - 3\beta mq^3$$
$$-66\gamma mq^3 + 2\alpha q^3 + 2\beta q^3 + 2\gamma q^3)].$$

3.3 Third family

Let

$$\begin{aligned} a &= \frac{8(\alpha+\beta+\gamma)^2(4\alpha - 10\beta + 25\gamma)}{453789}. \\ b &= \frac{1}{882}\left(-16\alpha^2 + 22\alpha\beta - 23\alpha\gamma + 38\beta^2 + 31\beta\gamma - 7\gamma^2\right). \\ c &= \frac{8\alpha^2 + 2\alpha\beta + 107\alpha\gamma - 6\beta^2 + 93\beta\gamma + 99\gamma^2}{1029}. \\ d &= -\frac{2\left(2\alpha^2 + 4\alpha\beta - 31\alpha\gamma + 2\beta^2 - 31\beta\gamma - 33\gamma^2\right)}{1029}. \end{aligned} \tag{38}$$

We make the transformation

$$u(x,t) = \frac{441}{2(\alpha+\beta+\gamma)}\partial_{x,x}\log\left(1 + \exp\left(kx - k^7t\right) + \rho\exp\left(2kx - 2k^7t\right)\right) \tag{39}$$

to obtain the soliton solution

$$u_{\text{soliton}}(x,t) = \frac{3528k^2\left(16e^{k^7t-kx} + e^{kx-k^7t} + 4\right)}{(\alpha+\beta+\gamma)\left(16e^{k^7t-kx} + e^{kx-k^7t} + 16\right)^2}\ \text{for}\,\rho = \frac{1}{16}. \tag{40}$$

A cnoidal wave solution is

$$u(x,t) = p + \frac{3mp}{1-2m}\text{cn}^2\left(\sqrt{\frac{2p(\alpha+\beta+\gamma)}{147(1-2m)}}(x-\lambda t) + \xi_0, m\right), \tag{41}$$

where

$$\lambda = -\frac{4(m-2)(m+1)p^3(48\alpha - 50\beta - 99\gamma)(\alpha+\beta+\gamma)^2}{3176523(2m-1)^2}. \tag{42}$$

Let us investigate the existence of two soliton solutions in the ansatz form

$$u(x,t) = \frac{441}{2(\alpha+\beta+\gamma)}\partial_{xx}\log(1 + \exp\theta_1 + \exp\theta_2 + \frac{1}{16}\,[\exp(2\theta_1) + \exp(2\theta_2)] +$$
$$\rho[\exp\,(\theta_1 + 2\theta_2) + \exp\,(2\theta_1 + \theta_2)] + \kappa\exp\,(\theta_1+\theta_2) + \rho^2\exp(2\theta_1 + 2\theta_2)),$$

where $\theta_1 = k_1x - k_1^7t$ and $\theta_2 = k_2x - k_2^7t$.

We have three soliton solutions for the following choices:

$$a = \frac{4\gamma^3}{147}, \ b = \frac{5\gamma^2}{14}, \ c = \frac{9\gamma^2}{7}, \ d = \frac{2\gamma^2}{7}, \ \alpha = 6\gamma, \ \beta = \frac{7\gamma}{2}. \tag{43}$$

$$\kappa = \frac{2k_1^4 - k_2^2k_1^2 + 2k_2^4}{2(k_1+k_2)^2\left(k_1^2 + k_2k_1 + k_2^2\right)}, \quad \rho = \frac{(k_1-k_2)^2\left(k_1^2 - k_2k_1 + k_2^2\right)}{16(k_1+k_2)^2\left(k_1^2 + k_2k_1 + k_2^2\right)} \tag{44}$$

Letting $\gamma = 42$ gives the Kaup–Kuperschmidt seventh-order equation:

$$u_t + 2016u^3u_x + 630u_x^3 + 2268u_xu_{2x} + 504u^2\,u_{3x} + 252u_{2x}u_{3x} + 147u_xu_{4x} + 42uu_{5x} + u_{7x} = 0. \tag{45}$$

3.4 Three soliton solutions

The three soliton solutions have the form

$$u_{3-\text{soliton}}(x,t) = 1/2\partial_{xx}\log\left(1 + \sum_{i,j,l=0}^{2} \rho_{i,j,l}\exp\left(i\left(k_1x - k_1^7t\right) + j\left(k_2x - k_2^7t\right) + l\left(k_3x - k_3^7t\right)\right)\right), \tag{46}$$

where $\rho_{i,j,l} = 1$ when $i + j + l = 1$. The parameter values are obtained as follows. First, we set

$$k_1x - k_1^7t = \log(z_1), k_2x - k_2^7t = \log(z_2) \text{ and } k_3x - k_3^7t = \log(z_3) \tag{47}$$

to get

$$\begin{gathered} u_t + 2016u^3u_x + 630u_x^3 + 2268u_xu_{2x} + 504u^2\,u_{3x} \\ +252u_{2x}u_{3x} + 147u_xu_{4x} + 42u_{5x} + u_{7x} = \Psi(z_1, z_2, z_3). \end{gathered}$$

Next, we solve the equation

$$\frac{\partial^{i+j+l}}{\partial z_1^i \partial z_2^j \partial^l z_3}\Psi(z_1, z_2, z_3)\big|_{z_1=z_2=z_3=0} = 0 \tag{48}$$

for $\rho_{i,j,l}$. The parameter values are:

$$\rho_{0,0,2} = \frac{1}{16}, \rho_{0,1,1} = \frac{2k_2^4 - k_3^2k_2^2 + 2k_3^4}{2(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)}, \rho_{0,1,2} = \frac{(k_2-k_3)^2\left(k_2^2 - k_3k_2 + k_3^2\right)}{16(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)}.$$

$$\rho_{0,2,0} = \frac{1}{16}, \rho_{0,2,1} = \frac{(k_2-k_3)^2\left(k_2^2 - k_3k_2 + k_3^2\right)}{16(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)}, \rho_{1,0,1} = \frac{2k_1^4 - k_3^2k_1^2 + 2k_3^4}{2(k_1+k_3)^2\left(k_1^2 + k_3k_1 + k_3^2\right)}.$$

$$\rho_{1,0,2} = \frac{(k_1-k_3)^2\left(k_1^2 - k_3k_1 + k_3^2\right)}{16(k_1+k_3)^2\left(k_1^2 + k_3k_1 + k_3^2\right)}, \rho_{1,1,0} = \frac{2k_1^4 - k_2^2k_1^2 + 2k_2^4}{2(k_1+k_2)^2\left(k_1^2 + k_2k_1 + k_2^2\right)}, \rho_{1,2,0} = \frac{(k_1-k_2)^2\left(k_1^2 - k_2k_1 + k_2^2\right)}{16(k_1+k_2)^2\left(k_1^2 + k_2k_1 + k_2^2\right)}.$$

$$\rho_{2,0,0}=\frac{1}{16},\rho_{2,0,1}=\frac{(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)}{16(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)},\rho_{2,1,0}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)}{16(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)}.$$

$$\rho_{0,2,2}=\frac{(k_2-k_3)^4\left(k_2^2-k_3k_2+k_3^2\right)^2}{256(k_2+k_3)^4\left(k_2^2+k_3k_2+k_3^2\right)^2}.$$

$$\rho_{1,1,1}=\frac{1}{4(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}$$

$$\times\ [4k_2^4k_1^8+4k_3^4k_1^8-2k_2^2k_3^2k_1^8-2k_2^6k_1^6-2k_3^6k_1^6-k_2^2k_3^4k_1^6-k_2^4k_3^2k_1^6+4k_2^8k_1^4+4k_3^8k_1^4$$
$$-k_2^2k_3^6k_1^4-6k_2^4k_3^4k_1^4-k_2^6k_3^2k_1^4-2k_2^2k_3^8k_1^2-k_2^4k_3^6k_1^2-k_2^6k_3^4k_1^2-2k_2^8k_3^2k_1^2+4k_2^4k_3^8-2k_2^6k_3^6+4k_2^8k_3^4].$$

$$\rho_{1,1,2}=\frac{\left(2k_1^4-k_2^2k_1^2+2k_2^4\right)(k_1-k_3)^2(k_2-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}.$$

$$\rho_{1,2,1}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)\left(2k_1^4-k_3^2k_1^2+2k_3^4\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}.$$

$$\rho_{1,2,2}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_3)^2(k_2-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)^2}{256(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)^2}.$$

$$\rho_{2,0,2}=\frac{(k_1-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)^2}{256(k_1+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)^2}.$$

$$\rho_{2,1,1}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)\left(2k_2^4-k_3^2k_2^2+2k_3^4\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}.$$

$$\rho_{2,1,2}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_3)^4(k_2-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)^2\left(k_2^2-k_3k_2+k_3^2\right)}{256(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^4(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)^2\left(k_2^2+k_3k_2+k_3^2\right)}.$$

$$\rho_{2,2,0}=\frac{(k_1-k_2)^4\left(k_1^2-k_2k_1+k_2^2\right)^2}{256(k_1+k_2)^4\left(k_1^2+k_2k_1+k_2^2\right)^2}.$$

$$\rho_{2,2,1}=\frac{(k_1-k_2)^4\left(k_1^2-k_2k_1+k_2^2\right)^2(k_1-k_3)^2(k_2-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)}{256(k_1+k_2)^4\left(k_1^2+k_2k_1+k_2^2\right)^2(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}.$$

$$\rho_{2,2,2}=\frac{(k_1-k_2)^4\left(k_1^2-k_2k_1+k_2^2\right)^2(k_1-k_3)^4(k_2-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)^2\left(k_2^2-k_3k_2+k_3^2\right)^2}{4096(k_1+k_2)^4\left(k_1^2+k_2k_1+k_2^2\right)^2(k_1+k_3)^4(k_2+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)^2\left(k_2^2+k_3k_2+k_3^2\right)^2}.$$

3.5 Four soliton solutions

The four soliton solutions have the form $u(x,t)=1/2\partial_{xx}\log f(x,t)$, where

$$\begin{cases} f(x,t) = \sum_{i_1=0}^{2}\sum_{i_2=0}^{2}\sum_{i_3=0}^{2}\sum_{i_4=0}^{2} B_{i_1i_2i_3i_4} z_1^{i_1} z_2^{i_2} z_3^{i_3} z_4^{i_4}, \\ z_j = \exp\left(k_j x - k_j^7 x\right) \text{ for any} j \end{cases} \tag{49}$$

The 76 coefficients $B_{i_1i_2i_3i_4}$ are given by

$$B_{0002} = \frac{1}{16}, B_{0011} = \frac{2k_3^4 - k_4^2k_3^2 + 2k_4^4}{2(k_3+k_4)^2\left(k_3^2 + k_4k_3 + k_4^2\right)}.$$

$$B_{0012} = \frac{(k_3-k_4)^2\left(k_3^2 - k_4k_3 + k_4^2\right)}{16(k_3+k_4)^2\left(k_3^2 + k_4k_3 + k_4^2\right)}, B_{0020} = \frac{1}{16}.$$

$$B_{0021} = \frac{(k_3-k_4)^2\left(k_3^2 - k_4k_3 + k_4^2\right)}{16(k_3+k_4)^2\left(k_3^2 + k_4k_3 + k_4^2\right)}.$$

$$B_{0022} = \frac{(k_3-k_4)^4\left(k_3^2 - k_4k_3 + k_4^2\right)^2}{256(k_3+k_4)^4\left(k_3^2 + k_4k_3 + k_4^2\right)^2}.$$

$$B_{0101} = \frac{2k_2^4 - k_4^2k_2^2 + 2k_4^4}{2(k_2+k_4)^2\left(k_2^2 + k_4k_2 + k_4^2\right)}.$$

$$B_{0102} = \frac{(k_2-k_4)^2\left(k_2^2 - k_4k_2 + k_4^2\right)}{16(k_2+k_4)^2\left(k_2^2 + k_4k_2 + k_4^2\right)}.$$

$$B_{0110} = \frac{2k_2^4 - k_3^2k_2^2 + 2k_3^4}{2(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)}.$$

$$B_{0111} = [4k_3^4k_2^8 + 4k_4^4k_2^8 - 2k_3^2k_4^2k_2^8 - 2k_3^6k_2^6 - 2k_4^6k_2^6 - k_3^2k_4^4k_2^6$$

$$-k_3^4k_4^2k_2^6 + 4k_3^8k_2^4 + 4k_4^8k_2^4 - k_3^2k_4^6k_2^4 - 6k_3^4k_4^4k_2^4$$

$$-k_3^6k_4^2k_2^4 - 2k_3^2k_4^8k_2^2 - k_3^4k_4^6k_2^2 - k_3^6k_4^4k_2^2 - 2k_3^8k_4^2k_2^2$$

$$+4k_3^4k_4^8 - 2k_3^6k_4^6 + 4k_3^8k_4^4]/[4(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)(k_2+k_4)^2$$

$$(k_3+k_4)^2\left(k_2^2 + k_4k_2 + k_4^2\right)\left(k_3^2 + k_4k_3 + k_4^2\right)].$$

$$B_{0112} = \frac{\left(2k_2^4 - k_3^2k_2^2 + 2k_3^4\right)(k_2-k_4)^2(k_3-k_4)^2\left(k_2^2 - k_4k_2 + k_4^2\right)\left(k_3^2 - k_4k_3 + k_4^2\right)}{32(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)(k_2+k_4)^2(k_3+k_4)^2\left(k_2^2 + k_4k_2 + k_4^2\right)\left(k_3^2 + k_4k_3 + k_4^2\right)}$$

$$B_{0120} = \frac{(k_2-k_3)^2\left(k_2^2 - k_3k_2 + k_3^2\right)}{16(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)}$$

$$B_{0121} = \frac{(k_2-k_3)^2\left(k_2^2 - k_3k_2 + k_3^2\right)(k_3-k_4)^2\left(k_3^2 - k_4k_3 + k_4^2\right)\left(2k_2^4 - k_4^2k_2^2 + 2k_4^4\right)}{32(k_2+k_3)^2\left(k_2^2 + k_3k_2 + k_3^2\right)(k_2+k_4)^2(k_3+k_4)^2\left(k_2^2 + k_4k_2 + k_4^2\right)\left(k_3^2 + k_4k_3 + k_4^2\right)}$$

$$B_{0122}=\frac{(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)(k_2-k_4)^2(k_3-k_4)^4\left(k_2^2-k_4k_2+k_4^2\right)\left(k_3^2-k_4k_3+k_4^2\right)^2}{256(k_2+k_3)^2\left(k_2^2+k_3k_2+k_3^2\right)(k_2+k_4)^2(k_3+k_4)^4\left(k_2^2+k_4k_2+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)^2}$$

$$B_{0200}=\frac{1}{16},\ B_{0201}=\frac{(k_2-k_4)^2\left(k_2^2-k_4k_2+k_4^2\right)}{16(k_2+k_4)^2\left(k_2^2+k_4k_2+k_4^2\right)}$$

$$B_{0202}=\frac{(k_2-k_4)^4\left(k_2^2-k_4k_2+k_4^2\right)^2}{256(k_2+k_4)^4\left(k_2^2+k_4k_2+k_4^2\right)^2},B_{0210}=\frac{(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)}{16(k_2+k_3)^2\left(k_2^2+k_3k_2+k_3^2\right)}.$$

$$B_{0211}=\frac{(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)(k_2-k_4)^2\left(k_2^2-k_4k_2+k_4^2\right)\left(2k_3^4-k_4^2k_3^2+2k_4^4\right)}{32(k_2+k_3)^2\left(k_2^2+k_3k_2+k_3^2\right)(k_2+k_4)^2(k_3+k_4)^2\left(k_2^2+k_4k_2+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)}.$$

$$B_{0212}=\frac{(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)(k_2-k_4)^4(k_3-k_4)^2\left(k_2^2-k_4k_2+k_4^2\right)^2\left(k_3^2-k_4k_3+k_4^2\right)}{256(k_2+k_3)^2\left(k_2^2+k_3k_2+k_3^2\right)(k_2+k_4)^4(k_3+k_4)^2\left(k_2^2+k_4k_2+k_4^2\right)^2\left(k_3^2+k_4k_3+k_4^2\right)}.$$

$$B_{0220}=\frac{(k_2-k_3)^4\left(k_2^2-k_3k_2+k_3^2\right)^2}{256(k_2+k_3)^4\left(k_2^2+k_3k_2+k_3^2\right)^2}-$$

$$B_{0221}=\frac{(k_2-k_3)^4\left(k_2^2-k_3k_2+k_3^2\right)^2(k_2-k_4)^2(k_3-k_4)^2\left(k_2^2-k_4k_2+k_4^2\right)\left(k_3^2-k_4k_3+k_4^2\right)}{256(k_2+k_3)^4\left(k_2^2+k_3k_2+k_3^2\right)^2(k_2+k_4)^2(k_3+k_4)^2\left(k_2^2+k_4k_2+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)}.$$

$$B_{0222}=\frac{(k_2-k_3)^4\left(k_2^2-k_3k_2+k_3^2\right)^2(k_2-k_4)^4(k_3-k_4)^4\left(k_2^2-k_4k_2+k_4^2\right)^2\left(k_3^2-k_4k_3+k_4^2\right)^2}{4096(k_2+k_3)^4\left(k_2^2+k_3k_2+k_3^2\right)^2(k_2+k_4)^4(k_3+k_4)^4\left(k_2^2+k_4k_2+k_4^2\right)^2\left(k_3^2+k_4k_3+k_4^2\right)^2}.$$

$$B_{1001}=\frac{2k_1^4-k_4^2k_1^2+2k_4^4}{2(k_1+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)},B_{1002}=\frac{(k_1-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)}{16(k_1+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)}.$$

$$B_{1010}=\frac{2k_1^4-k_3^2k_1^2+2k_3^4}{2(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)}$$

$$\begin{aligned}B_{1011}=&[4k_3^4k_1^8+4k_4^4k_1^8-2k_3^2k_4^2k_1^8-2k_3^6k_1^6-2k_4^6k_1^6-k_3^2k_4^4k_1^6\\&-k_3^4k_4^2k_1^6+4k_3^8k_1^4+4k_4^8k_1^4-k_3^2k_4^6k_1^4-6k_3^4k_4^4k_1^4-k_3^6k_4^2k_1^4-2k_3^2k_4^8k_1^2\\&-k_3^4k_4^6k_1^2-k_3^6k_4^4k_1^2-2k_3^8k_4^2k_1^2+4k_3^4k_4^8-2k_3^6k_4^6+4k_3^8k_4^4]\\&/[4(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)(k_1+k_4)^2(k_3+k_4)^2\\&\left(k_1^2+k_4k_1+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)].\end{aligned}$$

$$B_{1012}=\frac{\left(2k_1^4-k_3^2k_1^2+2k_3^4\right)(k_1-k_4)^2(k_3-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)\left(k_3^2-k_4k_3+k_4^2\right)}{32(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)(k_1+k_4)^2(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)}$$

$$B_{1020}=\frac{(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)}{16(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)}$$

$$B_{1021}=\frac{(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)(k_3-k_4)^2\left(k_3^2-k_4k_3+k_4^2\right)\left(2k_1^4-k_4^2k_1^2+2k_4^4\right)}{32(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)(k_1+k_4)^2(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)}$$

$$B_{1022}=\frac{(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)(k_1-k_4)^2(k_3-k_4)^4\left(k_1^2-k_4k_1+k_4^2\right)\left(k_3^2-k_4k_3+k_4^2\right)^2}{256(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)(k_1+k_4)^2(k_3+k_4)^4\left(k_1^2+k_4k_1+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)^2}$$

$$B_{1100}=\frac{2k_1^4-k_2^2k_1^2+2k_2^4}{2(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)}$$

$$B_{1101}=[4k_2^4k_1^8+4k_4^4k_1^8-2k_2^2k_4^2k_1^8-2k_2^6k_1^6-2k_4^6k_1^6-k_2^2k_4^4k_1^6-k_2^4k_4^2k_1^6$$
$$+4k_2^8k_1^4+4k_4^8k_1^4-k_2^2k_4^6k_1^4-6k_2^4k_4^4k_1^4-k_2^6k_4^2k_1^4-2k_2^2k_4^8k_1^2-k_2^4k_4^6k_1^2-$$
$$k_2^6k_4^4k_1^2-2k_2^8k_4^2k_1^2+4k_2^4k_4^8-2k_2^6k_4^6+4k_2^8k_4^4]/[4(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)$$
$$(k_1+k_4)^2(k_2+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)].$$

$$B_{1102}=\frac{\left(2k_1^4-k_2^2k_1^2+2k_2^4\right)(k_1-k_4)^2(k_2-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)\left(k_2^2-k_4k_2+k_4^2\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_4)^2(k_2+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)}$$

$$B_{1120}=\frac{\left(2k_1^4-k_2^2k_1^2+2k_2^4\right)(k_1-k_3)^2(k_2-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}$$

$$B_{1121}=(k_1-k_3)^2(k_2-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)(k_3-k_4)^2\left(k_3^2-k_4k_3+k_4^2\right)$$
$$(4k_2^4k_1^8+4k_4^4k_1^8-2k_2^2k_4^2k_1^8-2k_2^6k_1^6-2k_4^6k_1^6-k_2^2k_4^4k_1^6-k_2^4k_4^2k_1^6+4k_2^8k_1^4+4k_4^8k_1^4+$$
$$-k_2^2k_4^6k_1^4-6k_2^4k_4^4k_1^4-k_2^6k_4^2k_1^4-2k_2^2k_4^8k_1^2-k_2^4k_4^6k_1^2-k_2^6k_4^4k_1^2-2k_2^8k_4^2k_1^2+4k_2^4k_4^8-2k_2^6k_4^6+4k_2^8k_4^4)$$
$$/[64(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)(k_1+k_4)^2$$
$$(k_2+k_4)^2(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)].$$

$$B_{1122}=\left(2k_1^4-k_2^2k_1^2+2k_2^4\right)(k_1-k_3)^2(k_2-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)(k_1-k_4)^2$$
$$(k_2-k_4)^2(k_3-k_4)^4\left(k_1^2-k_4k_1+k_4^2\right)\left(k_2^2-k_4k_2+k_4^2\right)\left(k_3^2-k_4k_3+k_4^2\right)^2$$
$$/[512(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)$$
$$(k_1+k_4)^2(k_2+k_4)^2(k_3+k_4)^4\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)^2].$$

$$B_{1200}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)}{16(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)}$$

$$B_{1201}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_2-k_4)^2\left(k_2^2-k_4k_2+k_4^2\right)\left(2k_1^4-k_4^2k_1^2+2k_4^4\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_4)^2(k_2+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)}$$

$$B_{1202}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_4)^2(k_2-k_4)^4\left(k_1^2-k_4k_1+k_4^2\right)\left(k_2^2-k_4k_2+k_4^2\right)^2}{256(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_4)^2(k_2+k_4)^4\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)^2}$$

$$B_{1210}=\frac{(k_{1=}-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)\left(2k_1^4-k_3^2k_1^2+2k_3^4\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}$$

$$B_{1211}=(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)(k_2-k_4)^2\left(k_2^2-k_4k_2+k_4^2\right)$$
$$\Big(4k_3^4k_1^8+4k_4^4k_1^8-2k_3^2k_4^2k_1^8-2k_3^6k_1^6-2k_4^6k_1^6-k_3^2k_4^4k_1^6-k_3^4k_4^2k_1^6+4k_3^8k_1^4+4k_4^8k_1^4-k_3^2k_4^6k_1^4$$
$$-6k_3^4k_4^4k_1^4-k_3^6k_4^2k_1^4-2k_3^2k_4^8k_1^2-k_3^4k_4^6k_1^2-k_3^6k_4^4k_1^2-2k_3^8k_4^2k_1^2+4k_3^4k_4^8-2k_3^6k_4^6+4k_3^8k_4^4\Big)$$
$$/[64(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)$$
$$(k_1+k_4)^2(k_2+k_4)^2(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)].$$

$$B_{1212}=[(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_2-k_3)^2\left(k_2^2-k_3k_2+k_3^2\right)\left(2k_1^4-k_3^2k_1^2+2k_3^4\right)$$
$$(k_1-k_4)^2(k_2-k_4)^4(k_3-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)\left(k_2^2-k_4k_2+k_4^2\right)^2\left(k_3^2-k_4k_3+k_4^2\right)]$$
$$/[512(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)$$
$$(k_1+k_4)^2(k_2+k_4)^4(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)^2\left(k_3^2+k_4k_3+k_4^2\right)].$$

$$B_{1220}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_3)^2(k_2-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)^2}{256(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)^2}.$$

$$B_{1221}=[(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_3)^2(k_2-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)^2$$
$$(k_2-k_4)^2(k_3-k_4)^2\left(k_2^2-k_4k_2+k_4^2\right)\left(k_3^2-k_4k_3+k_4^2\right)\left(2k_1^4-k_4^2k_1^2+2k_4^4\right)]$$
$$/[512(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)^2$$
$$(k_1+k_4)^2(k_2+k_4)^2(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)].$$

$$B_{1222}=(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_3)^2(k_2-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)\left(k_2^2-k_3k_2+k_3^2\right)^2$$
$$(k_1-k_4)^2(k_2-k_4)^4(k_3-k_4)^4\left(k_1^2-k_4k_1+k_4^2\right)\left(k_2^2-k_4k_2+k_4^2\right)^2\left(k_3^2-k_4k_3+k_4^2\right)^2$$
$$/[4096(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)^2$$
$$(k_1+k_4)^2(k_2+k_4)^4(k_3+k_4)^4\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)^2\left(k_3^2+k_4k_3+k_4^2\right)^2].$$

$$B_{2000}=\frac{1}{16},B_{2001}=\frac{(k_1-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)}{16(k_1+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)},B_{2002}=\frac{(k_1-k_4)^4\left(k_1^2-k_4k_1+k_4^2\right)^2}{256(k_1+k_4)^4\left(k_1^2+k_4k_1+k_4^2\right)^2}.$$

$$B_{2010}=\frac{(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)}{16(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)}.$$

$$B_{2011}=\frac{(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)(k_1-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)\left(2k_3^4-k_4^2k_3^2+2k_4^4\right)}{32(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)(k_1+k_4)^2(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)}.$$

$$B_{2012}=\frac{(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)(k_1-k_4)^4(k_3-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)^2\left(k_3^2-k_4k_3+k_4^2\right)}{256(k_1+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)(k_1+k_4)^4(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)^2\left(k_3^2+k_4k_3+k_4^2\right)}.$$

$$B_{2020}=\frac{(k_1-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)^2}{256(k_1+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)^2}.$$

$$B_{2021}=\frac{(k_1-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)^2(k_1-k_4)^2(k_3-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)\left(k_3^2-k_4k_3+k_4^2\right)}{256(k_1+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)^2(k_1+k_4)^2(k_3+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_3^2+k_4k_3+k_4^2\right)}.$$

$$B_{2022}=\frac{(k_1-k_3)^4\left(k_1^2-k_3k_1+k_3^2\right)^2(k_1-k_4)^4(k_3-k_4)^4\left(k_1^2-k_4k_1+k_4^2\right)^2\left(k_3^2-k_4k_3+k_4^2\right)^2}{4096(k_1+k_3)^4\left(k_1^2+k_3k_1+k_3^2\right)^2(k_1+k_4)^4(k_3+k_4)^4\left(k_1^2+k_4k_1+k_4^2\right)^2\left(k_3^2+k_4k_3+k_4^2\right)^2}.$$

$$B_{2100}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)}{16(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)}.$$

$$B_{2101}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)\left(2k_2^4-k_4^2k_2^2+2k_4^4\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_4)^2(k_2+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)\left(k_2^2+k_4k_2+k_4^2\right)}.$$

$$B_{2102}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_4)^4(k_2-k_4)^2\left(k_1^2-k_4k_1+k_4^2\right)^2\left(k_2^2-k_4k_2+k_4^2\right)}{256(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_4)^4(k_2+k_4)^2\left(k_1^2+k_4k_1+k_4^2\right)^2\left(k_2^2+k_4k_2+k_4^2\right)}.$$

$$B_{2110}=\frac{(k_1-k_2)^2\left(k_1^2-k_2k_1+k_2^2\right)(k_1-k_3)^2\left(k_1^2-k_3k_1+k_3^2\right)\left(2k_2^4-k_3^2k_2^2+2k_3^4\right)}{32(k_1+k_2)^2\left(k_1^2+k_2k_1+k_2^2\right)(k_1+k_3)^2(k_2+k_3)^2\left(k_1^2+k_3k_1+k_3^2\right)\left(k_2^2+k_3k_2+k_3^2\right)}.$$

DOI: http://dx.doi.org/10.5772/intechopen.1004789

$$B_{2111} = [(k_1-k_2)^2(k_1^2-k_2k_1+k_2^2)(k_1-k_3)^2(k_1^2-k_3k_1+k_3^2)(k_1-k_4)^2 (k_1^2-k_4k_1+k_4^2)(4k_3^4k_2^8+4k_4^4k_2^8-2k_3^2k_4^2k_2^8-2k_3^6k_2^6-2k_4^6k_2^6-k_3^2k_4^4k_2^6 -k_3^4k_4^2k_2^6+4k_3^8k_2^4+4k_4^8k_2^4-k_3^2k_4^6k_2^4-6k_3^4k_4^4k_2^4-k_3^6k_4^2k_2^4-2k_3^2k_4^8k_2^2 -k_3^4k_4^6k_2^2-k_3^6k_4^4k_2^2-2k_3^8k_4^2k_2^2+4k_3^4k_4^8-2k_3^6k_4^6+4k_3^8k_4^4)] /[64(k_1+k_2)^2(k_1^2+k_2k_1+k_2^2)(k_1+k_3)^2(k_2+k_3)^2(k_1^2+k_3k_1+k_3^2) (k_2^2+k_3k_2+k_3^2)(k_1+k_4)^2(k_2+k_4)^2(k_3+k_4)^2(k_1^2+k_4k_1+k_4^2) (k_2^2+k_4k_2+k_4^2)(k_3^2+k_4k_3+k_4^2)].$$

$$B_{2112} = [(k_1-k_2)^2(k_1^2-k_2k_1+k_2^2)(k_1-k_3)^2(k_1^2-k_3k_1+k_3^2) (2k_2^4-k_3^2k_2^2+2k_3^4)(k_1-k_4)^4(k_2-k_4)^2(k_3-k_4)^2(k_1^2-k_4k_1+k_4^2)^2 (k_2^2-k_4k_2+k_4^2)(k_3^2-k_4k_3+k_4^2)]/[512(k_1+k_2)^2(k_1^2+k_2k_1+k_2^2)(k_1+k_3)^2 (k_2+k_3)^2(k_1^2+k_3k_1+k_3^2)(k_2^2+k_3k_2+k_3^2)(k_1+k_4)^4(k_2+k_4)^2(k_3+k_4)^2 (k_1^2+k_4k_1+k_4^2)^2(k_2^2+k_4k_2+k_4^2)(k_3^2+k_4k_3+k_4^2)].$$

$$B_{2120} = \frac{(k_1-k_2)^2(k_1^2-k_2k_1+k_2^2)(k_1-k_3)^4(k_2-k_3)^2(k_1^2-k_3k_1+k_3^2)^2(k_2^2-k_3k_2+k_3^2)}{256(k_1+k_2)^2(k_1^2+k_2k_1+k_2^2)(k_1+k_3)^4(k_2+k_3)^2(k_1^2+k_3k_1+k_3^2)^2(k_2^2+k_3k_2+k_3^2)}.$$

$$B_{2121} = [(k_1-k_2)^2(k_1^2-k_2k_1+k_2^2)(k_1-k_3)^4(k_2-k_3)^2(k_1^2-k_3k_1+k_3^2)^2 (k_2^2-k_3k_2+k_3^2)(k_1-k_4)^2(k_3-k_4)^2(k_1^2-k_4k_1+k_4^2)(k_3^2-k_4k_3+k_4^2) (2k_2^4-k_4^2k_2^2+2k_4^4)]/[512(k_1+k_2)^2(k_1^2+k_2k_1+k_2^2)(k_1+k_3)^4(k_2+k_3)^2 (k_1^2+k_3k_1+k_3^2)^2(k_2^2+k_3k_2+k_3^2)(k_1+k_4)^2(k_2+k_4)^2(k_3+k_4)^2 (k_1^2+k_4k_1+k_4^2)(k_2^2+k_4k_2+k_4^2)(k_3^2+k_4k_3+k_4^2)].$$

$$B_{2122} = [(k_1-k_2)^2(k_1^2-k_2k_1+k_2^2)(k_1-k_3)^4(k_2-k_3)^2(k_1^2-k_3k_1+k_3^2)^2 (k_2^2-k_3k_2+k_3^2)(k_1-k_4)^4(k_2-k_4)^2(k_3-k_4)^4(k_1^2-k_4k_1+k_4^2)^2 (k_2^2-k_4k_2+k_4^2)(k_3^2-k_4k_3+k_4^2)^2]/[4096(k_1+k_2)^2(k_1^2+k_2k_1+k_2^2)(k_1+k_3)^4 (k_2+k_3)^2(k_1^2+k_3k_1+k_3^2)^2(k_2^2+k_3k_2+k_3^2)(k_1+k_4)^4(k_2+k_4)^2(k_3+k_4)^4 (k_1^2+k_4k_1+k_4^2)^2(k_2^2+k_4k_2+k_4^2)(k_3^2+k_4k_3+k_4^2)^2].$$

$$B_{2200} = \frac{(k_1-k_2)^4(k_1^2-k_2k_1+k_2^2)^2}{256(k_1+k_2)^4(k_1^2+k_2k_1+k_2^2)^2}.$$

$$B_{2201} = \frac{(k_1-k_2)^4(k_1^2-k_2k_1+k_2^2)^2(k_1-k_4)^2(k_2-k_4)^2(k_1^2-k_4k_1+k_4^2)(k_2^2-k_4k_2+k_4^2)}{256(k_1+k_2)^4(k_1^2+k_2k_1+k_2^2)^2(k_1+k_4)^2(k_2+k_4)^2(k_1^2+k_4k_1+k_4^2)(k_2^2+k_4k_2+k_4^2)}$$

$$B_{2202} = \frac{(k_1-k_2)^4(k_1^2-k_2k_1+k_2^2)^2(k_1-k_4)^4(k_2-k_4)^4(k_1^2-k_4k_1+k_4^2)^2(k_2^2-k_4k_2+k_4^2)^2}{4096(k_1+k_2)^4(k_1^2+k_2k_1+k_2^2)^2(k_1+k_4)^4(k_2+k_4)^4(k_1^2+k_4k_1+k_4^2)^2(k_2^2+k_4k_2+k_4^2)^2}$$

$$B_{2210} = \frac{(k_1-k_2)^4(k_1^2-k_2k_1+k_2^2)^2(k_1-k_3)^2(k_2-k_3)^2(k_1^2-k_3k_1+k_3^2)(k_2^2-k_3k_2+k_3^2)}{256(k_1+k_2)^4(k_1^2+k_2k_1+k_2^2)^2(k_1+k_3)^2(k_2+k_3)^2(k_1^2+k_3k_1+k_3^2)(k_2^2+k_3k_2+k_3^2)}$$

$$B_{2211} = [(k_1 - k_2)^4 \left(k_1^2 - k_2k_1 + k_2^2\right)^2 (k_1 - k_3)^2 (k_2 - k_3)^2 \left(k_1^2 - k_3k_1 + k_3^2\right)$$
$$\left(k_2^2 - k_3k_2 + k_3^2\right)(k_1 - k_4)^2 (k_2 - k_4)^2 \left(k_1^2 - k_4k_1 + k_4^2\right)\left(k_2^2 - k_4k_2 + k_4^2\right)$$
$$\left(2k_3^4 - k_4^2k_3^2 + 2k_4^4\right)]/[512(k_1 + k_2)^4 \left(k_1^2 + k_2k_1 + k_2^2\right)^2 (k_1 + k_3)^2 (k_2 + k_3)^2$$
$$\left(k_1^2 + k_3k_1 + k_3^2\right)\left(k_2^2 + k_3k_2 + k_3^2\right)(k_1 + k_4)^2 (k_2 + k_4)^2 (k_3 + k_4)^2$$
$$\left(k_1^2 + k_4k_1 + k_4^2\right)\left(k_2^2 + k_4k_2 + k_4^2\right)\left(k_3^2 + k_4k_3 + k_4^2\right)].$$

$$B_{2212} = [(k_1 - k_2)^4 \left(k_1^2 - k_2k_1 + k_2^2\right)^2 (k_1 - k_3)^2 (k_2 - k_3)^2 \left(k_1^2 - k_3k_1 + k_3^2\right)$$
$$\left(k_2^2 - k_3k_2 + k_3^2\right)(k_1 - k_4)^4 (k_2 - k_4)^4 (k_3 - k_4)^2 \left(k_1^2 - k_4k_1 + k_4^2\right)^2$$
$$\left(k_2^2 - k_4k_2 + k_4^2\right)^2 \left(k_3^2 - k_4k_3 + k_4^2\right)]/[4096(k_1 + k_2)^4 \left(k_1^2 + k_2k_1 + k_2^2\right)^2 (k_1 + k_3)^2$$
$$(k_2 + k_3)^2 \left(k_1^2 + k_3k_1 + k_3^2\right)\left(k_2^2 + k_3k_2 + k_3^2\right)(k_1 + k_4)^4 (k_2 + k_4)^4 (k_3 + k_4)^2$$
$$\left(k_1^2 + k_4k_1 + k_4^2\right)^2 \left(k_2^2 + k_4k_2 + k_4^2\right)^2 \left(k_3^2 + k_4k_3 + k_4^2\right)].$$

$$B_{2220} = \frac{(k_1 - k_2)^4 \left(k_1^2 - k_2k_1 + k_2^2\right)^2 (k_1 - k_3)^4 (k_2 - k_3)^4 \left(k_1^2 - k_3k_1 + k_3^2\right)^2 \left(k_2^2 - k_3k_2 + k_3^2\right)^2}{4096(k_1 + k_2)^4 \left(k_1^2 + k_2k_1 + k_2^2\right)^2 (k_1 + k_3)^4 (k_2 + k_3)^4 \left(k_1^2 + k_3k_1 + k_3^2\right)^2 \left(k_2^2 + k_3k_2 + k_3^2\right)^2}.$$

$$B_{2221} = [(k_1 - k_2)^4 \left(k_1^2 - k_2k_1 + k_2^2\right)^2 (k_1 - k_3)^4 (k_2 - k_3)^4 \left(k_1^2 - k_3k_1 + k_3^2\right)^2$$
$$\left(k_2^2 - k_3k_2 + k_3^2\right)^2 (k_1 - k_4)^2 (k_2 - k_4)^2 (k_3 - k_4)^2 \left(k_1^2 - k_4k_1 + k_4^2\right)$$
$$\left(k_2^2 - k_4k_2 + k_4^2\right)\left(k_3^2 - k_4k_3 + k_4^2\right)]/[4096(k_1 + k_2)^4 \left(k_1^2 + k_2k_1 + k_2^2\right)^2 (k_1 + k_3)^4$$
$$(k_2 + k_3)^4 \left(k_1^2 + k_3k_1 + k_3^2\right)^2 \left(k_2^2 + k_3k_2 + k_3^2\right)^2 (k_1 + k_4)^2 (k_2 + k_4)^2 (k_3 + k_4)^2$$
$$\left(k_1^2 + k_4k_1 + k_4^2\right)\left(k_2^2 + k_4k_2 + k_4^2\right)\left(k_3^2 + k_4k_3 + k_4^2\right)].$$

$$B_{2222} = [(k_1 - k_2)^4 \left(k_1^2 - k_2k_1 + k_2^2\right)^2 (k_1 - k_3)^4 (k_2 - k_3)^4 \left(k_1^2 - k_3k_1 + k_3^2\right)^2$$
$$\left(k_2^2 - k_3k_2 + k_3^2\right)^2 (k_1 - k_4)^4 (k_2 - k_4)^4 (k_3 - k_4)^4 \left(k_1^2 - k_4k_1 + k_4^2\right)^2$$
$$\left(k_2^2 - k_4k_2 + k_4^2\right)^2 \left(k_3^2 - k_4k_3 + k_4^2\right)^2]/[65536(k_1 + k_2)^4 \left(k_1^2 + k_2k_1 + k_2^2\right)^2 (k_1 + k_3)^4$$
$$(k_2 + k_3)^4 \left(k_1^2 + k_3k_1 + k_3^2\right)^2 \left(k_2^2 + k_3k_2 + k_3^2\right)^2 (k_1 + k_4)^4 (k_2 + k_4)^4 (k_3 + k_4)^4$$
$$\left(k_1^2 + k_4k_1 + k_4^2\right)^2 \left(k_2^2 + k_4k_2 + k_4^2\right)^2 \left(k_3^2 + k_4k_3 + k_4^2\right)^2].$$

For a graphical illustration, see **Figure 4**.

3.6 *N*-soliton solutions

The *N*-soliton solutions have the form $u(x,t) = 1/2\partial_{xx} \log f(x,t)$, where

$$\begin{cases} f(x,t) = \sum_{i_1,i_2,\cdots,i_N=0}^{2} B_{i_1i_2\cdots i_N} \prod_{j=1}^{N} z_j^{i_j}, \\ z_j = \exp\left(k_jx - k_j^7x\right) \text{ for any} j \end{cases} \tag{50}$$

In order to find the unknown coefficients $B_{i_1i_2\cdots i_N}$, we define

$$\Psi(z_1, z_2, \cdots, z_N) = u_t + 2016u^3u_x + 630u_x^3 + 2268u_xu_{2x} + 504u^2\, u_{3x}$$
$$+252u_{2x}u_{3x} + 147u_xu_{4x} + 42uu_{5x} + u_{7x}.$$

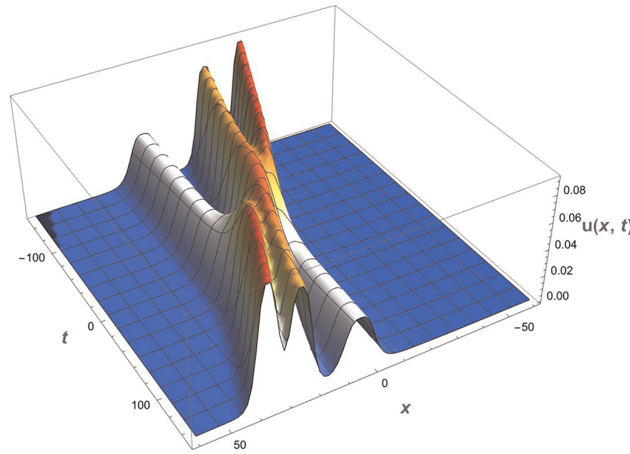

Figure 4.
Four soliton solution: $\{k_1 = 0.489591, k_2 = 0.68479, k_3 = 0.104676, k_4 = 0.733527\}$.

The number $B_{i_1 i_2 \cdots i_N}$ is obtained from the equation

$$\frac{\partial^{i_1+i_2+\cdots+i_N}}{\partial z_1^{i_1} \partial z_2^{i_2} \cdots \partial z_N^{i_n}} \Psi(0,0,0, \ldots, 0) = 0 \tag{51}$$

4. Bilinearization

Let us consider the case when $d = c/2 - b$. The family to be considered is

$$u_t + au^3 u_x + bu_x^3 + cuu_x u_{2x} + (c/2 - b)u^2\, u_{3x} + \alpha u_{2x} u_{3x} + \beta u_x u_{4x} + \gamma uu_{5x} + u_{7x} = 0. \tag{52}$$

Let

$$u(x,t) = A\partial_{xx} f(x,t). \tag{53}$$

Plugging the ansatz in Eq. (53) into Eq. (52) and integrating once with respect to x, taking a zero integration constant gives

$$\begin{aligned}
&3f_x(8f^4(4Af_{xxx}^2(Ab + 5\beta - 5\gamma)D^2{}_x(f \cdot f) + 2f(A(\gamma - \beta)f_{xxxxxxx}D^2{}_x(f \cdot f) + 2f(D^8{}_x(f \cdot f)+\\
&D_{xt}(f \cdot f))) + (A(\alpha - \beta + \gamma) - 140)D^4{}_x(f \cdot f)^2 + (A(\beta + \gamma) - 112)D^2{}_x(f \cdot f)D^6{}_x(f \cdot f))\\
&+D^2{}_x(f \cdot f)^4\left(aA^3 + 6A(12\alpha + 4Ab - 2Ac + 3\beta + 77\gamma) - 20160\right)\\
&+4f^2D^2{}_x(f \cdot f)^2D^4{}_x(f \cdot f)\left(A^2(c - 2b) - 3A(4\alpha + \beta + 19\gamma) + 3360\right))\\
&-48Af^2f_x^4f_{xxx}(4Ab + 15(\gamma - \beta))D^2{}_x(f \cdot f) - 24Af^2f_x^2f_{xxx}(3(4Ab - 15\beta + 15\gamma)D^2{}_x(f \cdot f)^2\\
&+10f^2(\beta - \gamma)D^4{}_x(f \cdot f)) + 8Af_x^3(2f^2(\beta - \gamma)(f^2(20f_{xxx}^2 - 2ff_{xxxxxxx} + D^6{}_x(f \cdot f))\\
&+15D^2{}_x(f \cdot f)D^4{}_x(f \cdot f)) + 9(3Ab + 20(\gamma - \beta))D^2{}_x(f \cdot f)^3) + 4Af^2(\beta - \gamma)f_{xxx}(-4f^4(20f_{xxx}^2 + D^6{}_x(f \cdot f))\\
&+8f^5f_{xxxxxxx} - 45D^2{}_x(f \cdot f)^3 + 30f^2D^2{}_x(f \cdot f)D^4{}_x(f \cdot f)) + 288Af^2(\beta - \gamma)f_x^6f_{xxx}\\
&+96Af_x^7(Ab + 12(\gamma - \beta))D^2{}_x(f \cdot f) + 48Af_x^5(3(2Ab - 15\beta + 15\gamma)D^2{}_x(f \cdot f)^2\\
&+5f^2(\beta - \gamma)D^4{}_x(f \cdot f)) + 288A(\gamma - \beta)f_x^9 = 0.
\end{aligned}$$

The choices

$$\left\{\alpha = \frac{5\gamma}{2}, \beta = \gamma, a = \frac{15\gamma^3}{784}, b = 0, c = \frac{15\gamma^2}{28}\right\} \tag{54}$$

will give the following bilinear form

$$D^1_{xt}(f \cdot f) + D^8_x(f \cdot f) = 0 \tag{55}$$

This corresponds to the KdV7 $(A = 2, \gamma = 28)$

$$u_t + 420u^3u_x + 420u_xu_{2x} + 210u^2\, u_{3x} + 70u_{2x}u_{3x} + 28u_xu_{4x} + 28uu_{5x} + u_{7x} = 0. \tag{56}$$

This KdV7 admits one and two soliton solutions. However, it does not have three solitons solutions despite the fact that it admits bilinear form.

One soliton solution: $u(x,t) = 2\partial_{xx}\log\left(1 + \exp\left(k_1x - k_1^7t\right)\right)$..

Two soliton solution:

$u(x,t) = 2\partial_{xx}\log\left(1 + \exp\left(k_1x - k_1^7t\right) + \exp\left(k_2x - k_2^7t\right) + A_{12}\exp\left(k_1x - k_1^7t\right)\exp\left(k_2x - k_2^7t\right)\right)$,

where

$$A_{1,2} = \frac{(k_2 - k_1)^2\left(k_1^2 - k_2k_1 + k_2^2\right)^2}{(k_1 + k_2)^2\left(k_1^2 + k_2k_1 + k_2^2\right)^2}. \tag{57}$$

Breather: $u(x,t) = 2\partial_{xx}\log\left(pe^{kx-\lambda t} + qe^{\lambda t - kx} + r\sin(\kappa x - \mu t)\right)$, where

$$\begin{aligned} \lambda &= k\left(-7\kappa^6 + k^6 - 21\kappa^2k^4 + 35\kappa^4k^2\right). \\ \mu &= \kappa\left(-\kappa^6 + 7k^6 - 35\kappa^2k^4 + 21\kappa^4k^2\right). \\ p &= -\frac{\kappa^2r^2\left(3\kappa^2 - k^2\right)^2}{4k^2q\left(\kappa^2 - 3k^2\right)^2}. \end{aligned} \tag{58}$$

Let us consider a more general than Eq. (56) KdV7

$$\begin{aligned} &u_t + 420u^3u_x + 420uu_xu_{xx} + 210u^2u_{xxx} + 70u_{xx}u_{xxx} + 28u_xu_{xxxx} \\ &+28uu_{xxxxx} + u_{7x} + 6puu_x + 45qu^2u_x + 15qu_xu_{xx} + pu_{xxx} + 15quu_{xxx} + quu_{5x} + u_{7x} = 0. \end{aligned}$$

This KdV7 admits the bilinear form

$$D^1_{xt}(f \cdot f) + pD^4_x(f \cdot f) + qD^6_x(f \cdot f) + D^8_x(f \cdot f) = 0. \tag{59}$$

The one soliton solutions are

$$u(x,t) = \frac{k^2e^{k^3t\left(k^4 + k^2q + p\right) + kx}}{\left(e^{k^3t\left(k^4 + k^2q + p\right)} + e^{kx}\right)^2}.$$

The two soliton solutions are

$$u(x,t) = \partial_{xx}\log(1+\exp(k_1x-w_1t)+\exp(k_2x-w_2t)+A_{1,2}\exp(k_1x-w_1t)\exp(k_2x-w_2t)),$$

where $w_1 = k_1^3p + k_1^5q + k_1^7, w_2 = k_2^3p + k_2^5q + k_2^7,$ and

$$A_{1,2} = \frac{(k_1-k_2)^2\left(5k_1^2q - 5k_2k_1q + 5k_2^2q + 7k_1^4 - 14k_2k_1^3 + 21k_2^2k_1^2 - 14k_2^3k_1 + 7k_2^4 + 3p\right)}{(k_1+k_2)^2\left(5k_1^2q + 5k_2k_1q + 5k_2^2q + 7k_1^4 + 14k_2k_1^3 + 21k_2^2k_1^2 + 14k_2^3k_1 + 7k_2^4 + 3p\right)}. \tag{60}$$

On the other hand, direct calculations show that the KdV7

$$u_t + \frac{(3\alpha-5\beta)(2\beta+\gamma)^2}{1176}u^3u_x + \frac{1}{56}(\beta-\gamma)(2\beta+\gamma)u_x^3 + \frac{1}{28}(2\beta+\gamma)(2\alpha-3\beta+3\gamma)uu_xu_{xx}$$
$$+\frac{1}{28}(2\beta+\gamma)(\alpha-2\beta+2\gamma)u^2u_{xxx} + \alpha u_{xx}u_{xxx} + \beta u_xu_{xxxx} + \gamma uu_{xxxxx} + u_{xxxxxxx} = 0$$

may be written in the following Hirota's bilinear form [4]:

$$\begin{cases} 2D_{xt}(f\cdot f) + \dfrac{7(\beta-\gamma)}{2\beta+\gamma}D_x^4(f\cdot g) - \dfrac{3(\beta-3\gamma)}{2\beta+\gamma}D_x^4(f\cdot f) + \dfrac{14(6\alpha-8\beta-7\gamma)}{2\beta+\gamma}D_x^8(f\cdot f) \\ +\dfrac{14(6\alpha-8\beta-7\gamma)}{2\beta+\gamma}(g\cdot g) = 0. \\ D_x^4(f\cdot f) - (f\cdot g) = 0. \\ u(x,t) = A\partial_{xx}\log f(x,t), A = \dfrac{168}{\gamma+2\beta}. \end{cases} \tag{61}$$

The seventh-order Kaup-Kuperschmidt Eq. (4) belongs to this class ($A = 1/2$). Using the obtained bilinear form, we may obtain all the results we presented in previous sections (for the special case $d = c/2 - b$).

5. Forced KdV7

The forced KdV7 is written as

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2\,u_{3x} + \alpha u_{2x}u_{3x} + \beta u_xu_{4x} + \gamma u_{5x} + u_{7x} = f(t). \tag{62}$$

The forced Sawada-Kotera-Ito Eq. (2) and the forced Lax Eq. (3) admit the exact solution.

$$u_t + 252u^3u_x + 63u_x^3 + 378u_xu_{2x} + 126u^2\,u_{3x} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = f(t).$$

Exact solution:

$$u(x,t) = B + 2\operatorname{sech}^2(x-\lambda(t)) + F(t), \tag{63}$$

where

$$\lambda(t) = 4\int\left(63B^3 + 189B^2F(t) + 126B^2 + 189BF(t)^2 + 252BF(t) + 84B + 63F(t)^3 + 126F(t)^2 + 84F(t) + 16\right)dt. \tag{64}$$

and

$$F(t) = \int f(t). \tag{65}$$

$$u_t + 140u^3u_x + 70u_x^3 + 280u_xu_{2x} + 70u^2\,u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0.$$

- Exact solution:

$$u(x,t) = B + 2\text{sech}^2(x - \lambda(t)) + F(t), \tag{66}$$

where

$$\lambda(t) = 4\int\begin{pmatrix} 35B^3 + 105B^2F(t) + 70B^2 + 105BF(t)^2 + 140BF(t) \\ +56B + 35F(t)^3 + 70F(t)^2 + 56F(t) + 16 \end{pmatrix} dt. \tag{67}$$

and

$$F(t) = \int f(t). \tag{68}$$

For other parameter values, we obtained the following result:
If $\alpha + \beta + \gamma \neq 0$ and

$$\begin{gathered} a = \frac{1}{63}d(\alpha + \beta + \gamma) \\ b = \frac{1}{126}\left(-\alpha^2 + 4\alpha\beta - 11\alpha\gamma + 5\beta^2 - 5\beta\gamma - 10\gamma^2 + 126d\right) \\ c = \frac{1}{21}\left(5\alpha\gamma + 5\beta\gamma + 5\gamma^2 - 42d\right) \end{gathered} \tag{69}$$

then the forced KdV7 in Eq. (2) admits the exact solution

$$u(x,t) = B + \frac{252}{\alpha + \beta + \gamma}\text{sech}^2(x - \lambda(t)) + F(t), \tag{70}$$

where

$$\lambda(t) = \frac{1}{63}\int\begin{pmatrix} F(t)\left(3\alpha B^2d + 3\beta B^2d + 3B^2\gamma d + 504Bd + 1008\gamma\right) \\ +F(t)^2(3\alpha Bd + 3\beta Bd + 3B\gamma d + 252d) + F(t)^3(\alpha d + \beta d + \gamma d) \end{pmatrix} dt$$

$$+\frac{1}{63}t\left(\alpha B^3d + \beta B^3d + B^3\gamma d + 252B^2d + 1008B\gamma + 4032\right),$$

$$F(t) = \int f(t).$$

See also [5].

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References

[1] Yao R-X, Li Z-B. Conservation Laws and new exact solutions for the generalized seventh order KdV equation. Chaos, Solitons and Fractals. 2004;**20**: 259-266. DOI: 10.1016/S0960-0779(03) 00373-4

[2] Fan E, Hona YC. Generalized Tanh method extended to special types of nonlinear equations. Zeitschrift fur Naturforschung A. 2002;**57**:692-700. DOI: 10.1515/zna-2002-0809

[3] Wazwaz AM. Soliton solutions for seventh-order Kawahara equation with time-dependent coefficients. Modern Physics Letters B. 2011;**25**:643-648. DOI: 10.1142/S0217984911026012

[4] Optical and Quantum Electronics. 2020;**52**:511. DOI: 10.1007/s11082-020-02628-7

[5] Alvaro H. Salas, computing exact solutions to a generalized lax seventh-order forced KdV equation (KdV7). Applied Mathematics and Computation. 2010;**216**(8):2333-2338