Chapter

An Improved Particle Swarm Optimization Method for Nonlinear Optimization

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Abstract

Nonlinear optimization is becoming challengeable in information sciences and various industrial applications, but nonlinear problems solved by the classical particle swarm-based methods are usually characterized with low efficiency, accuracy, and convergence speed in specific issues. To solve these problems and enhance the nonlinear optimization performance, an improved metaheuristic particle swarm optimization (PSO) model is proposed here. First, the optimization principles and model of the new method are introduced, and algorithms of the improved PSO are presented. Then, influence of the model parameters, input dimensions, and different nonlinear problems on the PSO optimization characterizations are studied by Pareto set solving and optimization performance comparison. The analysis regarding diverse nonlinear problems and optimization methods demonstrates the feasibility of the proposed method. Finally, the performance evaluation is exhibited by the case study of nonlinear parameter optimization, CEC benchmark problems, and rank sum test, which all verify its effectiveness and reliability, as well as the significance and great application promise. Additionally, the future work is discussed.

Keywords: particle swarm optimization (PSO), nonlinear optimization, performance evaluation, Pareto set, benchmark problem

1. Introduction

Nonlinear optimization problems including the constrained and unconstrained situations have drawn more and more attention, where the law of physics is usually formulated as the mathematical form and theories of maximum or minimum of the objective functions. Thus, optimal solutions can be obtained, and the performance of objective functions or the model to be optimized can be enhanced. The nonlinear optimization techniques may include the steepest decent, Newton method, conjugate gradient, trust region, non-gradient methods for unconstrained problems, and the basic dual, feasible point, penalty, augmented Lagrangian, and barrier methods for

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constrained problems [1], which have been widely applied in engineering, economics, portfolio management, medical diagnosis, computer science, and mathematics when the theories of statistics, approximation and operational research are considered [1, 2]. Typical applications and studies are reported in keyframe-based visual inertial slam nonlinear optimization [3], gas networks, beam lines, packing ellipsoids optimization and deep excavation analysis [4], as well as compressed sensing and parallel computing [5].

Except classical and common optimization methods, some new nonlinear optimization approaches and dynamic models are also proposed to solve the convex, facility layout, and limit analysis problems with fuzzy related constraints [6, 7]. Despite that the constraints are usually diverse in different optimization problems where continuously differentiable objective functions with bound and inequality constraints under uncertainty and binary variables are making the optimization problems more complicated [8], they can be well solved by using a multiparametric mixed-integer and sparse-based algorithm as well as linear programming, recurrent neural network, Bayesian optimization library, and direction interior-point techniques [9]. The result manifests that these methods can not only build a framework for dynamic optimization problems but also identify better integer solutions with promising feasibility [10]. Other strategies including the machine learning algorithms for hyperparameter optimization [11], pattern and stochastic search [12, 13] are also becoming a new trend in the combined optimization models, which are also demonstrated to be valid in solving parameter optimization and real-time nonlinear optimization problems.

Performance evaluation methods regarding nonlinear optimization are also investigated in related reports. Except the basic performance indicators such as the hypervolume, running time, and contribution, a large amount of evaluation methods and techniques are reported [14, 15]. Aiming at the robustness and convergence analysis, an approximation technique is presented [16], which employs the respective dual norm of the first-order derivatives of constraints to form two sparsity preserving ways for efficient computation of the derivative in solving large-scale and control problems [17]. As for the complexity and model structural stability analysis [18], various matrix conditioning and numerical analysis methods within a unifying framework of a proximity-scaling or a threshold technique [19] are applied, which have been presented as an effective example in process engineering. Nevertheless, human activities and related constraint conditions frequently affect the nonlinear optimization process. Solving nonlinear optimization problems more efficiently and with better models is still a challenge under limited resources, especially when large-scale and high-dimensional optimization applications are encountered in practice.

It is worth noting that the particle swarm optimization (PSO) method is one of the most important meta-heuristic optimization methods [1, 17] and has gained increasing attention in nonlinear problem solving, which can achieve the goal of optimization by simulating the behavior of particle swarms and searching for the best solution through updating the particle moving displacement and velocity. When an optimal PSO model is built, it can be applied in various fields and solve a variety of practical problems. For instance, the optimal design of the corona ring on high voltage composite insulators [20], optimal free-defect synthesis of four-bar linkage with joint clearance [21], and fast nonlinear predictive control on field programmable gate array (FPGA) can be achieved by PSO with dynamic population size [22]. Theoretically, constrained multivariable predictive controllers with model uncertainty and portfolio optimization problem [23], parameter optimization of nonlinear gray Bernoulli model [24], nonlinear regression, and unconstrained nonlinear optimization problems can also be well solved efficiently. However, the optimization efficiency and accuracy are usually influenced and limited by specific application situations and various constraints.

Consequently, diverse improved PSO models are proposed to alleviate the challenges mentioned above. For instance, to solve the mixed integer nonlinear optimization problem where the practical hydrothermal scheduling system with an inequality constraint treatment mechanism is highly complex, an improved PSO-based discrete binary and dynamic search-space squeezing strategy is proposed [25], the numerical calculation results compared with the traditional dynamic, nonlinear, and evolutionary programming methods show that the improved PSO behaves the best in terms of convergence speed, running time, and minimum cost. To identify the nonlinear system faster, more accurate, and choose the inertial weights and constriction coefficients more properly, modified PSO models for locating better local minimum are also proposed in [26]. Chaotic PSO [27, 28] and the enhanced PSO using exponential probability distribution testing results manifest their effectiveness in nonlinear optimization performance improvement [29, 30].

In recent years, aiming at specific applications and theoretical problems, numerous nonlinear optimization methods are proposed. A mixed logarithmic barrier-augmented Lagrangian method and GA algorithm for solving nonlinear optimization problems defined with fuzzy relational equations and max-Lukasiewicz composition are presented in [31], where the excellent computational performance is highlighted. Furthermore, a novel five-step discrete-time forms of Zhang dynamics (DTZD) and state-space decomposition algorithm [32] for time-varying nonlinear optimization, an approximation scheme [33] for distributionally robust nonlinear optimization, and a user preference enabling method [34] for constrained multi-objective nonlinear optimization are all reported. As for PSO-related nonlinear optimization methods, they are mainly focused on the combination of meta-heuristic optimization technologies, time-varying and nonlinear control system prediction. Such as the hybrid methods [35] are proposed to solve the nonlinear optimization problem evaluated by the reliability of a complex system, identification accuracy, and convergence speed, which have been efficiently used in various engineering fields, for instance, the parameter recognition for nonlinear dynamic hysteretic models [36] and industrial robot [37], dynamic wireless sensor network optimization [38] in data transfer acceleration and energy losses reduction, as well as the optimization of electromagnetic geophysical data [39]. Although some improved PSO models with Brownian Motion are reported [40, 41], they are usually separately applied in the optimization process, where the evolutionary mechanisms of PSO are unchanged. Such as, the Fractional Brownian Motion (FBM) model was proposed in [40] to forecast a non-stationary time series, where the Quantum-Behaved PSO (QPSO) is only utilized to optimize the multifractional hurst exponent, while the PSO and FBM are independently considered. In addition, the machine learning-based algorithms of support vector machine (SVM) and PSO combined algorithms are gradually considered in the precipitation data prediction [42], which is also the future research direction of nonlinear optimization. In other words, although tremendous advances in nonlinear optimization algorithms and software have been seen in recent decades, combining modern intelligent algorithms and platforms is the key to solving big data or large-scale problems with real-time strategies.

Motivated by the Euler-Maruyama (EM) principles and the optimization process of the PSO model, a nonlinear optimization method based on improved PSO algorithm is proposed here.

2. Models and algorithms

In this section, the basic theories of the improved PSO-based optimization models and principles are introduced in detail. The frameworks of the improved PSO and nonlinear optimization algorithms are also described and explained.

2.1 Models and principles

The traditional Brownian particle motion in the moment of t can be described as a standard normal distribution, and the distribution probability density is defined as follows:

$$N(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{x^2}{2\sigma^2 t}} \tag{1}$$

However, the random motion mode of these particle swarms is usually related to the searching efficiency, accuracy, convergence performance, and solving speed. Consequently, the Euler-Maruyama (EM) principle is considered to solve the particle motion equation, namely,

$$x(t) = x_0 + \int_0^t f(x(s))ds + \int_0^t \mathrm{g}(x(s))dw(s) \quad (0 \le t \le T) \tag{2}$$

where $x(t)$ is the solution of a stochastic differential equation (SDE), x_0 is the initial value of $x(t_0)$, f and g are the scalar functions, and w is a scalar standard Brownian motion within the domain of [0, T]. The differential form of (2) can be calculated by

$$\mathrm{d}x(t) = f(x(t))dt + g(x(t))dw(t) \quad (x(0) = x_0, 0 \le t \le T) \tag{3}$$

Let $\Delta \mathrm{t} = T/L$, $\tau_i = i\Delta t$, where L is the number of the time interval, which is a positive integer, and i is the evolution or iteration sequence. Thus, the SDE can be obtained as a discretized form of

$$x_i = x_{i-1} + f(x_{i-1})\Delta t + g(x_{i-1})\Delta w(t) \tag{4}$$

where the increment of $\Delta w(t)$ is defined as

$$\Delta w(t) = w(\tau_i) - w(\tau_{i-1}) \quad i = 1, 2, \ldots, L \tag{5}$$

The integral form of (4) can be computed and expressed as

$$x(\tau_\mathrm{i}) = x(\tau_\mathrm{i\text{-}1}) + \int_{\tau_\mathrm{i\text{-}1}}^{\tau_\mathrm{i}} f(x(s))ds + \int_{\tau_\mathrm{i\text{-}1}}^{\tau_\mathrm{i}} \mathrm{g}(x(s))dw(s) \tag{6}$$

Assuming that the scalar functions of $f(x)$ and $g(x)$ are expressed as the approximate forms of $f(x) = \lambda x$ and $g(x) = \mu x$, the differential forms of (6) can be obtained as follows:

$$dx(t) = \lambda x(t)dt + \mu x(t)dw(t) \tag{7}$$

where λ and μ are constants, and the solution of (7) is calculated as follows:

$$x(t) = x_0 \cdot e^{\left(\left(\lambda - \frac{\mu^2}{2}\right)t + \mu w(t)\right)} \tag{8}$$

If the time interval of Δt can be divided into a smaller form of δt by a new introduced constant of R, namely,

$$R = \frac{\Delta t}{\delta t} \tag{9}$$

the increment of $w(t)$ can be further expressed as a cumulative form of

$$\Delta w(t) = w(\tau_i) - w(\tau_{i-1}) = \sum_{k=iR-R+1}^{iR} dw_k \tag{10}$$

Combining the traditional PSO algorithms, the displacement and velocity of the moving particle of X_i and V_i can be depicted as

$$X_{\mathrm{i}} = X_i + V_i \tag{11}$$

$$V_i = V_i + c_1 r_1 \cdot (P_{best} - X_i) + c_2 r_2 \cdot (G_{best} - X_i) \tag{12}$$

where c_1 and c_2 are the constants, r_1 and r_2 are the random numbers, and P_{best} and G_{best} are the local best solution and global best solution in the optimization and iteration process. However, the optimization process is usually influenced by the random number of r_1 and r_2 greatly. If they are replaced by the EM solution of $x(t)$ expressed in (8) when (9) and (10) are considered, an improved PSO model can be obtained and expressed as follows:

$$\mathrm{r}_1 = r_2 = x_0 \cdot e^{\left(\left(\lambda - \frac{\mu^2}{2}\right)t + \mu w(t)\right)} \tag{13}$$

It can be found that the random optimization process within the lower and upper searching space are improved as a discretized and more efficient way, which is expected to enhance the optimization efficiency and accuracy by replacing the random optimization process with a more regular way when moving and updating these particle swarms.

2.2 Algorithms

Specifically, the framework of the improved PSO algorithm is expressed in Algorithm 1 as follows:

Algorithm 1: Improved PSO method.

1 Initialize the basic parameters of the particle population including the moving velocity V_i and position X_i, lower and upper bound on search space of X_{low} and X_{upper}, population number N_p, the maximum iteration number N, and set the initial best position as $P_{best} = X_i$;

2 Evaluate the moving state of each particle and set the global best position of $G_{best} = \min\{P_{best}(1), P_{best}(2), P_{best}(3), \ldots P_{best}(\mathrm{i})\}$;

3 **While** G_{best} is not found
4   **For** $i = 1$ to N
5   Calculate the particle moving state parameters of r_1 and r_2 according to (13);
6   Calculate the fitness value of each particle for the moving velocity and position;
7   **If** the fitness value of $F(X_i) < F(P_{best})$
8   Update the position with $P_{best} = X_i$;
9   **If** the fitness value of $F(P_{best}) < F(G_{best})$
10   Update the global best position with $G_{best} = P_{best}$;
11   **End if**
12   Update the moving velocity of V_i according to (12);
13   **End for**
14 **End while**
15   Obtain the final result of global best value G_{best};
16 **End**

Explanatorily, the moving position and speed of the particle swarm are first initialized randomly within the lower and upper bound. Then, the fitness value of each particle is calculated until the maximum iteration number is reached. Further, the position and velocity of the particles are compared and updated if the local best solution is searched, and the global best solution is consequently obtained by comparing the best local solutions in each iteration. Thereafter, the particle moving position and velocity can be updated combining (12) and (13). Finally, the global best value of the objective function can be obtained in the PSO when the termination condition is satisfied, and the circulative iteration is finished. When the improved PSO method is applied to solve nonlinear optimization problems, the framework is presented as the process of Algorithm 2.

Algorithm 2. Nonlinear optimization process by improved PSO method

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1 Set up the nonlinear problem with specific models;
2 Initialize the basic parameters in the nonlinear model;
3 Initialize the basic parameters of PSO models, including the maximum iteration number, lower and upper bounds of the search space, initial value of variables in the nonlinear problem;
4 While N < maximum iteration number
5   Initialize population and particle swarm;
6   Search for the global best nonlinear value at each iteration of particle evolution;
7   Update particle moving speeds and positions with (11) and (12) through the improved PSO;
8   Update the search direction with the optimal hyperparameters;
9   Evaluate the nonlinear model and compute the global best values;
10 Update the best solution for every nonlinear problem;
11 Obtain the nonlinear optimization result
12 End while

First, the constrained or unconstrained nonlinear problems should be modeled in a mathematic form. Then, the basic parameters of variables in the nonlinear model are initialized with starting values, as well as evolution parameters in the improved PSO, such as $\lambda = 0.1$, $\mu = 1$, $R = 4$, population = 300, $N = 30$, $C_1 = 0.5$, $C_2 = 0.5$, and the weight of ω = 0.7. Then, the nonlinear optimization is conducted within a circulative iteration, where the global best nonlinear values can be obtained after updating the particle moving positions and speeds in each of the evolution step. Consequently, the search direction with optimal hyperparameter is updated, and the nonlinear optimization result is evaluated until the best solution is calculated. Consequently, the new enhanced PSO algorithm can be utilized in nonlinear parameter optimization by improving the particle evolution and searching forms with better convergence performance.

3. Optimization characterization

In this section, the nonlinear optimization characterization is mainly investigated from the perspective of model parameter and dimensional influence. Besides, the nonlinear optimization method based on the improved PSO algorithm is preliminarily validated by different nonlinear optimization problems.

3.1 Influence of model parameters

When the improved PSO models are built, their optimization performance is usually influenced by model parameters including the constants of λ, μ, and R mentioned in (7) and (9), as well as the population size of the particle swarm, iteration number of N, C_1, and C_2. Typically, these parameters are set as λ = 0.1, μ = 1, R = 4, population = 300, N = 30, C_1 = 0.5, C_2 = 0.5, and the weight of ω = 0.7, when one of them is changed according to the parameter setting expressed in **Table 1**, other parameters remain unchanged. There are totally nine groups of variables for each of the optimization model parameter.

On the other hand, a typical nonlinear optimization problem based on Bohachevsky test function is mainly studied, and the objective function is defined as,

$$o(x) = {x_1}^2 + 2{x_2}^2 - 0.3\cos(3\pi x_1) - 0.4\cos(4\pi x_2) + 0.7 \tag{14}$$

This nonlinear function is investigated within the domain of $x_i \in [-100, 100]$ for all i = 1, 2, and there are two variables or dimensions in the objective function. When the global minimum of the nonlinear function is searched by the proposed improved PSO

Serial no.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
λ	0.01	0.05	0.1	0.2	0.5	1	2	5	10
μ	0.1	0.2	0.5	1	2	5	10	20	50
R	1	2	3	4	5	7	10	20	50
Population	50	100	200	300	400	500	600	800	1000
N	10	20	30	50	60	80	100	200	500
ω	0.01	0.1	0.2	0.3	0.5	0.6	0.7	0.8	1
C_1	0.01	0.05	0.1	0.2	0.3	0.5	0.6	0.8	1
C_2	0.01	0.05	0.1	0.2	0.3	0.5	0.6	0.8	1

Table 1.
Parameter setting in different comparing models.

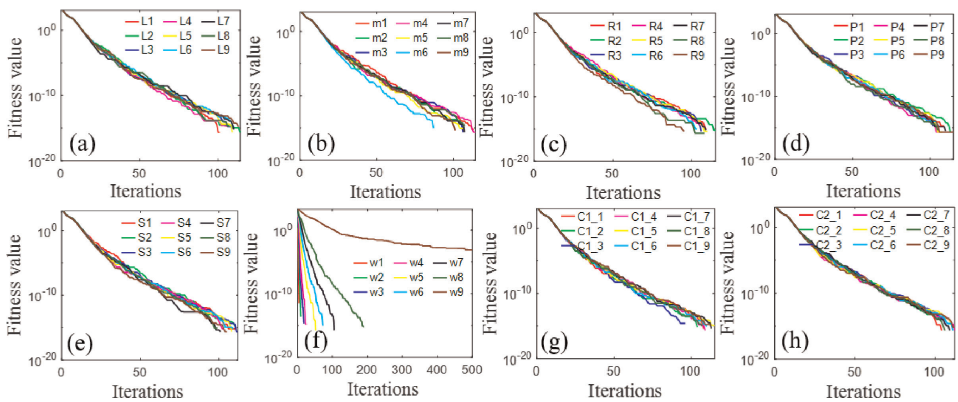

Figure 1.
Influence of eight model parameters to the algorithm convergence: (a) λ; (b) μ; (c) R*; (d) population size; (e)* N*; (f) ω; (g)* C_1*; and (h)* C_2.

based method, the convergence performance characterized by the fitness values in each of the iteration under different model parameters is illustrated and expressed in **Figure 1**. Explanatorily, the legends of *L1* to *L2* appeared in **Figure 1(a)** represent the parameter values of λ changed from 0.01 to 10 according to the second line mentioned in **Table 1**. Likewise, when the improved PSO model parameters of μ, R, population, N, ω, C_1 and C_2 change, the fitness values in each of the iterations and optimization process are calculated and presented in **Figure 1(b)**-**(h)**, respectively.

Obviously, all these fitness values are decreased gradually in the evolution process no matter how these parameters change. Moreover, the convergence speeds are almost consistent when the model coefficients of λ, μ, R, population, N, C_1 and C_2 change observing from **Figure 1(a)**-**(e)** and **Figure 1(g)** and **(h)**. In other words, these seven parameters are not very sensitive to the convergence of the improved PSO-based nonlinear optimization algorithm. However, the convergence speeds and optimization performance of the proposed method are greatly influenced by the parameter of the searching weight ω, which can be found from the computing results presented in **Figure 1(f)**. It is evident that, as ω gains from *w1* to *w9* (namely, from 0.01 to 1), the convergence speed gradually decreases, and the maximum iteration number to search for the best global solutions increases progressively.

3.2 Influence of input dimension

To study the influence of the input dimension of the variable on the solving process and the final solution results, four typical nonlinear and multi-objective optimization functions from dtlz test suite including dtlz1a, dtlz2a, dtlz4a and dtlz7a [43] are chosen and investigated. There are two or three objective functions in the nonlinear optimization for each of the test problem, and the input dimensions are set as 2, 4, 6, 8, and 10 in the five control groups. After calculating for these four dtlz test functions, five groups of Pareto set (PS) for each of the multi-objective and nonlinear problems can be obtained, respectively, as shown in **Figure 2**.

When the input dimension of d increases from 2 to 10, the PS calculation results for dtlz1a test problem keep nearly the consistent, which means that the nonlinear optimization and evolution process are immune to the change of the input dimension,

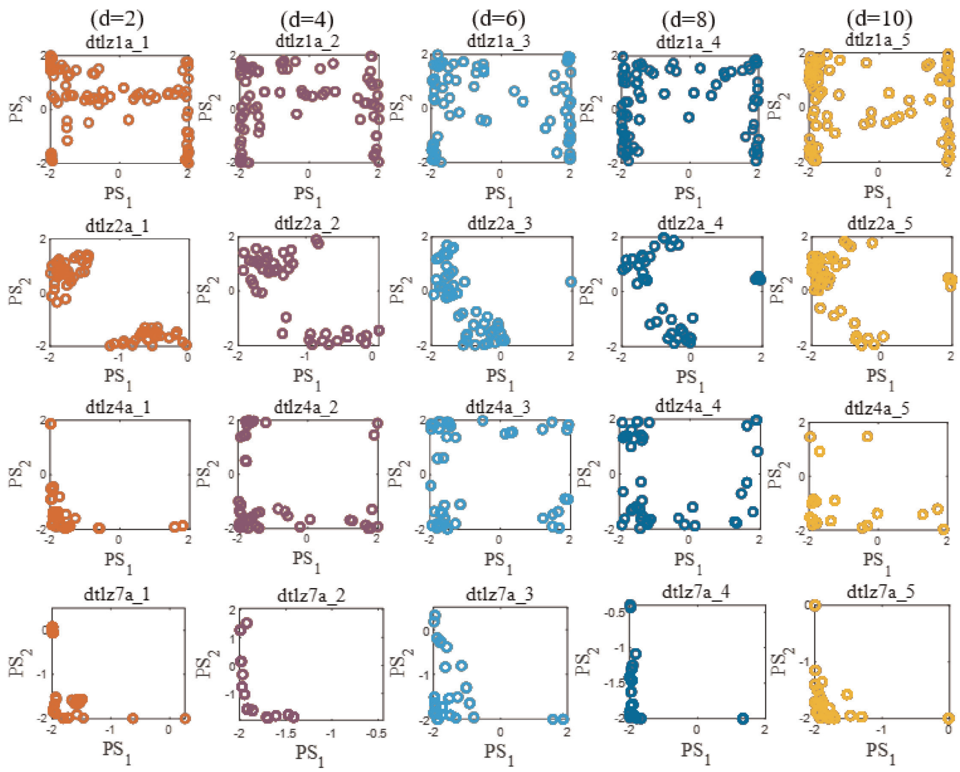

Figure 2.
Influence of the input dimension to Pareto set.

and the improved PSO-based nonlinear optimization method is robust and reliable in solving dtlz problems. Besides, the PS computing results acquired for the other dtlz problems of dtlz2a, dtlz4a, and dtlz7a also indicate that the optimal solutions can be well found when the input dimension gains from 2 to 10, and the distribution of these PS results are almost consistent. On all accounts, the proposed nonlinear algorithm can not only solve the dtlz problems but also can maintain a relative higher robustness and reliability when the input dimension changes.

3.3 Study on different nonlinear problems

What's more, eight different nonlinear problems are further investigated and studied with various output variables including the MMF1 to MMF3, kno1, oka1, oka2, vlmop2, and vlmop3 functions [44]. When the improved PSO-based nonlinear optimization method is applied to solve these problems, the calculation results of the Pareto set are obtained and expressed in **Figure 3**.

Generally, these nonlinear models have two main objective functions except vlmop3. Although the eight nonlinear problems are different in mathematic definition and global characterization, all their Pareto set can be acquired exactly. Namely, the optimization method is capable of solving various nonlinear problems especially for multi-objective models, while the robustness and reliability can always keep consistent regardless of the change of model parameters. Besides, the preliminary

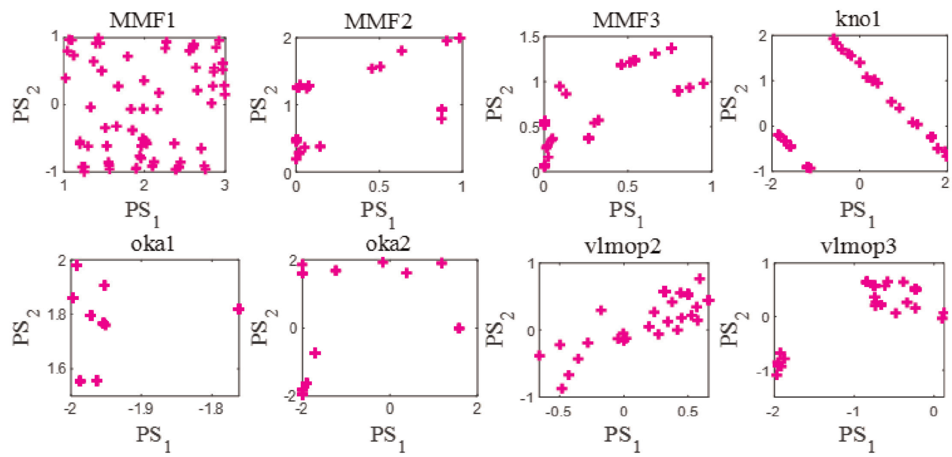

Figure 3.
Influence of different nonlinear problems on the new PSO calculating results.

calculation results of Pareto set through different nonlinear models also demonstrate its feasibility in nonlinear optimization and sophisticated problem solving.

4. Nonlinear optimization results

Based on the model parameter investigation and nonlinear optimization performance characterization, more nonlinear optimization results including the Pareto fronts (PFs) obtained from different nonlinear optimization methods and problems are studied and compared in this section. The advantages of the proposed improved PSO based method are further verified.

4.1 Comparison of Pareto fronts

First, the PFs computed for these 12 different nonlinear models including MMF1 to MMFL3, vlmop2, vlmop3, oka1, oka2, kno1, dtlz1a to dtlz7a as mentioned above are presented, where 12 groups of PFs calculation results by the new PSO-based nonlinear optimization algorithm are also presented and compared in **Figure 4**. Observed by the results from PF1 to PF12, it can also be found that almost all the concavity and convexity for these nonlinear models are well exhibited even for MMF2 and oka2. Namely, better nonlinear optimization results of PFs can be obtained from the new PSO-based algorithm with more efficient particle moving patterns relative to the traditional PSO method with random way.

4.2 Comparison of different methods

According to the principle of steepest decent algorithm (SDA), if the optimized nonlinear function of *f(X)* has first-order continuous partial derivative, after the iteration for k times, the searching direction for the global best solution is defined as follows:

$$S^k = -\nabla f\left(X^k\right) \tag{15}$$

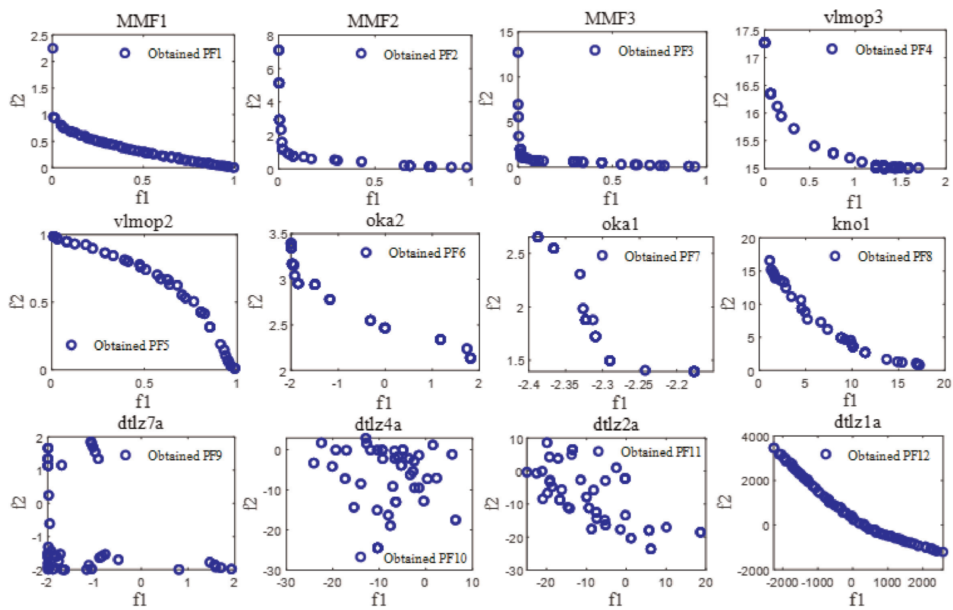

Figure 4.
PFs calculated from the new improved PSO model.

However, the convergence speed of the traditional SDA is slow, and the saw-tooth phenomenon will appear especially when the searching space is near the minimum value. As for the conjugate gradient algorithm (CGA), when an initial point of X^0 and a descent direction of S^0 is set, the first searching result of X^1 can be obtained as

$$\nabla f\left(X^1\right)^T S^0 = 0 \tag{16}$$

To overcome the saw-tooth phenomenon occurred in the steepest decent method, twice searching for a binary quadratic nonlinear equations is needed to find the minimum value. Then, a conjugate search direction vector for S^0 which satisfies the following equation is introduced:

$$\left(S^0\right)^T A S^1 = 0 \tag{17}$$

where A is a symmetric positive matrix and satisfies

$$f(X) = \frac{1}{2} X^T A X + b^T X + c \quad (\mathrm{b} \in E^n, \;\; \text{c is a constant}) \tag{18}$$

Consequently, the conjugate search direction of S^1 is obtained as follows:

$$S^1 = -\nabla f\left(X^1\right) + \frac{\left(S^0\right)^T A \nabla f\left(X^1\right)}{\left(S^0\right)^T A S^0} S^0 \tag{19}$$

if a binary quadratic nonlinear function is searched along S^0 with a random initial point of X^0 and then searched through the conjugate direction of S^1, the minimum solution of X^* can be acquired through CGA.

Besides, another classical variable metric method called Davidon-Fletcher-Powell algorithm (DFPA) is also applied and compared. When a scale matrix of ${H_k}^{-1}$ is considered and calculated as follows:

$$H_{k+1} = H_k + \frac{\Delta X^k \left(\Delta X^k\right)^T}{\left(\Delta X^k\right)^T \Delta g_k} - \frac{H_k \Delta g_k \cdot {\Delta g_k}^T H_k}{{\Delta g_k}^T H_k \Delta g_k} \tag{20}$$

where g_k and X_k are described as

$$\begin{cases} g_k = \nabla f\left(X^k\right) \\ \Delta g_k = g_{k+1} - g_k \\ \Delta X^k = X^{k+1} - X^k \end{cases} \tag{21}$$

Then, the searching direction of S_k can be obtained as

$$S_k = -H_k g_k \tag{22}$$

DFPA is efficient in nonlinear optimization when searching for the global minimum, but the round-off error and imprecise searching may lead to the existence of singularity for H_k. Thus, an improved method of Broyden-Fletcher-Goldfarb-Shanno algorithm (BFGSA) is proposed, where the positive definite matrix of H_k satisfies,

$$H_{k+1} = \frac{H_k + \left[\left(1 + \frac{{\Delta g_k}^T H_k \Delta g_k}{\left(\Delta X^k\right)^T \Delta g_k}\right) \Delta X^k \left(\Delta X^k\right)^T - H_k \Delta g_k \left(\Delta X^k\right)^T - \Delta X^k {\Delta g_k}^T H_k\right]}{\left(\Delta X^k\right)^T \Delta g_k} \tag{23}$$

Although BFGSA has similar mathematical features to that of the DFPA, the singular matrix will not be easily generated, and it has more stable numerical solutions and requires less searching precision than DFPA.

Furthermore, five groups of typical nonlinear optimization test functions are applied and tested, where the 2D sphere function (T1), Himmelblau's function (T2), Goldstein-Price function (T3), 2D Styblinski-Tang function (T4), and 5D Styblinski-Tang function (T5) are included, and the definition of these test problems is as follows:

$$\begin{cases} T1 = \sum_{i=1}^{n} {x_i}^2 \\ T2 = \left({x_1}^2 + x_2 - 11\right)^2 + \left(x_1 + {x_2}^2 - 7\right)^2 \\ T3 = \left[1 + (x_1 + x_2 + 1)^2 \cdot \left(19 - 14x_1 + 3{x_1}^2 - 14x_2 + 6x_1x_2 + 3{x_2}^2\right)\right] \\ \cdot \left[30 + (2x_1 - 3x_2)^2 \left(18 - 32x_1 + 12{x_1}^2 + 48x_2 - 36x_1x_2 + 27{x_2}^2\right)\right] \\ T4 = \frac{1}{2}\sum_{\mathrm{i}=1}^{2} \left({x_i}^4 - 16{x_i}^2 + 5x_i\right) \\ T5 = \frac{1}{2}\sum_{\mathrm{i}=1}^{5} \left({x_i}^4 - 16{x_i}^2 + 5x_i\right) \end{cases} \tag{24}$$

When these classical nonlinear optimization methods are applied to these functions test, the global best values and the optimized objective values, as well as the iteration numbers are acquired and shown in **Table 2**. Although the global best values for the test functions of T1, T2, T4, and T5 are accurately calculated with few iterations, there exist relative great optimization errors for the optimization results of T3.

Besides, the similar PSO algorithms expanded as PSO_2 and PSO_3 are also compared. Different with the traditional PSO algorithm (PSO_1), the PSO_2 is modeled with a cumulative form of r_i (i = 1, 2), for instance,

$$\mathrm{r}_1 = r_2 = \sum \sqrt{\frac{1}{N}} \cdot rand(N) \tag{25}$$

Likewise, the expanded model of PSO_3 referring to the reasoning process of the improved PSO is defined as

$$\mathrm{r}_1 = r_2 = x_0 \cdot e^{\left(\left(\lambda - \frac{\mu^2}{2}\right)t + \mu \cdot \sum \sqrt{\frac{1}{N}} \cdot rand(N)\right)} \tag{26}$$

Except the random term of r_i, the extended models of PSO_2 and PSO_3 have the same optimization and evolution process with that of the improved PSO as mentioned before in Algorithm 2. After calculating for the test functions of T1 to T5 by the comparative PSO_1 to PSO_3 and the proposed improved PSO method, the nonlinear optimization results are obtained and presented in **Table 3**.

Comparing the optimization results shown in **Tables 3** and **4**, the global minimum values of T1 to T3 can all be obtained exactly through these four meta-heuristic algorithms of PSO, especially for T3, while the classical four algorithms mentioned in **Table 3** fail to solve. Besides, T4 can also be solved well by PSO_1, PSO_3, and the improved PSO method. As for T5, PSO_1 and the improved PSO can all be optimized with relative smaller errors compared with the true global best values. In other words, the improved PSO method behaves the best among these four PSO models in the nonlinear optimization, which is even better than that of these four classical optimization methods mentioned in **Table 3**.

4.3 Comparison of different nonlinear problems

Different nonlinear problems are further studied in this section to validate the feasibility and reliability of the proposed improved PSO method. Typically, 12 nonlinear models including F1 to F12 are mainly investigated and tested. The butterfly optimization algorithm (BOA), the traditional PSO method, hybrid PSO with BOA (HPSOBOA), as well as the new improved PSO model (PSO_new) are applied and compared in the F series functions test. After the calculation and evolution in the nonlinear problem testing, the change trend of fitness values in the global minimum and best solution searching or iteration process are obtained, as expressed in **Figure 5**. It can be clearly seen from the fitness calculation results for F1, F2, F3, F4, F5 and F7, F9, F10, F12 that the new proposed PSO method has the fastest convergence speed and least iteration numbers in searching for the global best solution of these nine nonlinear problems.

As for the left F6, F8, and F11, the new PSO also behaves slightly better than that of the traditional PSO method. In brief, the new proposed PSO method not only has better nonlinear optimization performance than that of the BOA and HPSOBOA in

Function	Global Best value	SDA		CGA		DFPA		BFGSA	
		Iteration	Objective value	Iteration	Objective value	Iteration	Objective value	Iteration	Objective value
T1	0	1	0	2	0	1	0	1	0
T2	0	8	0	7	0	5	0	5	0
T3	**3**	22	**30**	7	**30**	5	**30**	5	**30**
T4	−78.33	1	−78.3323	2	−78.3323	1	−78.3323	1	−78.3323
T5	−195.83	1	−195.8308	2	−195.8308	1	−195.8308	1	−195.8308

Table 2.
Nonlinear optimization results from four classical methods.

Function	Global Best value	PSO_1		PSO_2		PSO_3		Improved PSO	
		Iteration	Objective value	Iteration	Objective value	Iteration	Objective value	Iteration	Objective value
T1	0	20	0	20	0.000152	20	0	9	0
T2	0	50	0	50	0	50	0	46	0
T3	3	50	3	50	3	50	3	39	3
T4	−78.33	50	−78.30	50	−78.20	50	−78.30	37	−78.30
T5	−195.83	25	−195.8257	25	−195.6255	25	−195.00	25	−195.8153

Table 3.
Nonlinear optimization results from different PSO models.

most cases but also features faster convergence speed and higher efficiency in searching for the global best solutions than that of the traditional PSO. Consequently, the feasibility and reliability of the improved PSO is validated, which is expected to be applied to more sophisticated nonlinear problem solving and optimization in practical applications.

5. Performance evaluation and application

In this section, the proposed improved PSO based nonlinear optimization method is mainly evaluated by the numerical experiments and performance indicators. Besides, comparisons regarding different PSO models and nonlinear problems are also conducted and investigated. Finally, a nonlinear parameter optimization case study, typical CEC benchmark problems, and rank sum test are also considered, which all demonstrate the feasibility and preferable performance of the proposed improved PSO method.

5.1 Experimental setup

The numerical experiments are mainly conducted through the platform of MATLAB R2022b with the computer configuration of Intel(R) Core (TM), i7-7700HQ CPU @ 3.2GHz and 16.0GB RAM, 64-bit Windows operating system. Besides, more than 12 nonlinear optimization problems including the MMF functions, vlmop, and dtlz series functions as mentioned above are tested. Four related PSO models are mainly applied in the nonlinear solution comparison, where the main parameters are set as λ = 0.1, μ = 1, R = 4, population = 300, N = 30, C_1 = 0.5, C_2 = 0.5, and the weight of ω = 0.7. Furthermore, all the numerical experiments are conducted for 50 times to guarantee repeatability and reliability.

5.2 Hypervolume and time

The hypervolume indicator is primarily used to measure the performance of the nonlinear optimization results, which is defined as follows [45]:

$$H\mathrm{v}(S) = \Lambda\left(\left\{q \in \mathbb{R}^d \mid \exists p \in S : p \leq q, q \leq r\right\}\right) \tag{27}$$

where S is the given point set and r is a reference point, and S is the region measure weekly dominated by the bound of r, and Λ is the Lebesgue measure. The traditional PSO method and the improved PSO models with different parameters are applied in the tests for 12 nonlinear problems. The optimization indicators of hypervolume (Hv) and running time (/s) of these algorithms calculated in each of the model are obtained and presented in **Table 4**, where the population number of the particle swarm and the coefficient of λ are annotated in the brackets.

It can be seen that all the parameters of λ are set as 0.1, while the population for the traditional PSO is 400 and diverse for the improved PSO. As the population increases from 200 to 400, the hypervolumes are also increased slightly in each of the test problems through the improved PSO method, while the running time changes a little. Namely, the robustness of the improved PSO method is increased while the optimization efficiency is unaffected as the population size gains. Noteworthily, when the

Problems	PSO (400, 0.1)		Improved PSO (200, 0.1)		Improved PSO (300, 0.1)		Improved PSO (400, 0.1)	
	Hv	Time	Hv	Time	Hv	Time	Hv	Time
MMF1	3.6438	2.7727	3.6372	2.9084	3.6274	2.7539	3.6401	2.6624
MMF2	3.4524	3.7696	3.5099	4.1256	3.4987	3.9866	3.5277	4.1519
MMF3	3.4532	4.4021	3.4052	3.9400	3.5574	3.8329	3.5124	4.0124
vlmop3	−26.663	3.4645	−26.678	3.1630	−26.8320	3.1953	−26.7590	3.5025
vlmop2	3.3231	2.9952	3.2624	3.7602	3.2790	4.2806	3.2945	3.8899
oka2	−2.2072	3.6441	−2.1605	3.8799	−2.4836	3.4863	−2.5338	3.3566
oka1	3.2775	8.4793	1.7238	7.3141	2.6560	8.1473	3.5655	8.2796
kno1	−61.9418	4.2376	−73.6724	5.1355	−91.1023	4.2069	−93.3501	4.6448
dtlz7a	14.6180	2.7965	12.5099	2.3893	15.8068	3.1955	14.6047	2.6112
dtlz4a	208.0310	2.9061	167.9735	2.3034	442.6884	2.2043	331.8512	2.3637
dtlz2a	397.6778	2.9043	41.0199	2.2796	355.5462	2.6709	292.8354	2.7049
dtlz1a	−4.211e6	1.3554	−6.917e6	1.16905	−7.558e6	1.1458	−8.836e6	1.2760

Table 4.
Hypervolume and running time calculation results.

population and λ are set as 300 and 0.1, most of the hypervolume is bigger than that of the traditional PSO, and the vast majority of running time is less than the traditional PSO. It manifests that the new proposed PSO is more capable of nonlinear optimization with higher calculation efficiency, stability, and robustness.

5.3 Comparison of nonlinear solutions

The optimization process is further studied in this section when four typical nonlinear test problems of T1 to T4 as defined and mentioned before in (24) are considered. Besides, four related PSO methods including the traditional particle swarm optimization method of PSO_1, expanded models of PSO_2 and PSO_3, as well as the new proposed improved PSO model of PSO_4 are applied and compared. The optimization and global searching process can be visualized and presented in **Figure 6**.

Observed by the nonlinear optimization process for T1, T3, and T4, it can be found that although the PSO_1 can search the accurate global best solutions, the searching space is limited and the particles usually gather within a small scope, which may lead to the leak of some local extreme points and important positions. As a contrast, the searching coverage from the improved PSO_4 is greatly extended, which makes it more efficient and accurate in finding the local best and global best values in the nonlinear optimization problems solving. However, relative bigger searching errors may be generated when the PSO_2 for T4 and PSO_3 for T2 are investigated. In a word, the feasibility of the improved PSO method with higher optimization efficiency and robustness is exhibited again from the perspective of optimization process observation and investigation.

5.4 Convergence analysis

The convergence of different PSO-based algorithms are also compared in this section; after the calculation and evolution for nonlinear test problems of T1 to T4, the solutions of objective values in each iteration are presented in **Figure** 7, where the legend of 'mean (Ti best)' (i = 1, 2, 3, 4) represents the mean of the best population value, while the legend of 'mean (Ti)' (i = 1, 2, 3, 4) denotes the mean of the current population value.

It can be found that the convergence curves for T1 to T3 by each of the corresponding PSO method are very alike, especially for the global best solution curves, which manifests that both PSO_2 and PSO_3 can solve the nonlinear optimization problems with a similar accuracy to PSO_1, as well as the improved PSO_4. Besides, no big differences are found among these PSO methods when observing the calculating results for T4, namely, all of them can search for the global best values within several iterations.

5.5 Wilcoxon rank sum test

Furthermore, the Wilcoxon rank sum test [46, 47] which recognizes the difference of two datasets is also considered to evaluate the benchmark problem-solving performance of these algorithms. Primarily, the comparison results indicated by binomial estimator, Hodges-Lehmann (HL) estimator, W value, Z value, mean, standard deviation (SD), as well as p value are calculated and presented in **Table** 5.

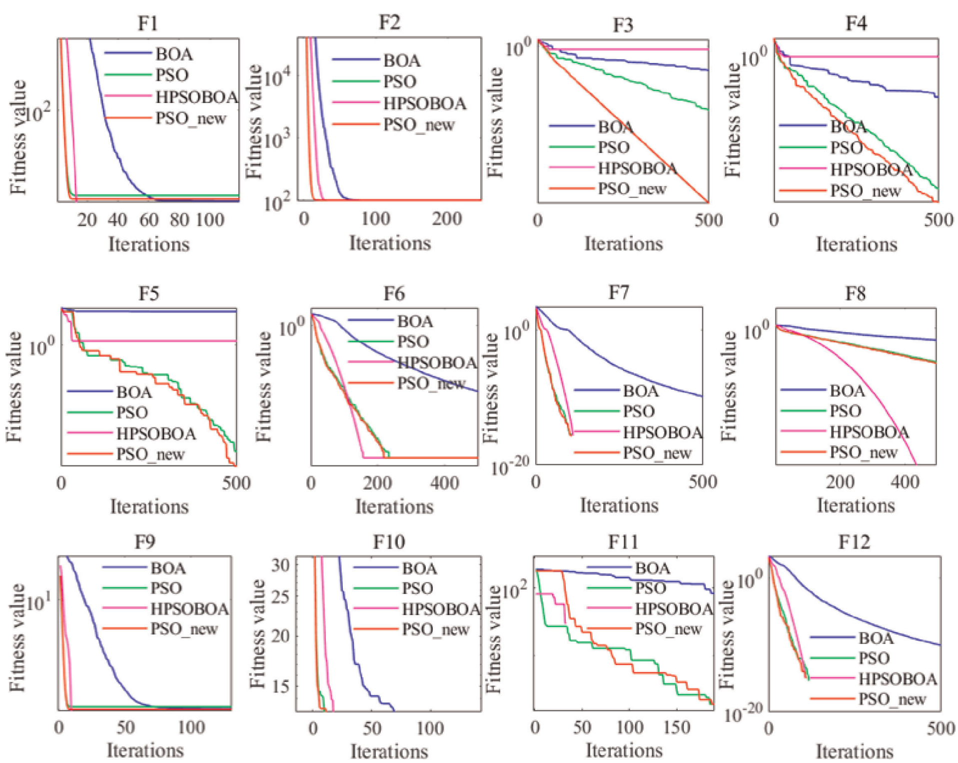

Figure 5.
Results of fitness for F functions.

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Indicator	PSO_4 vs	For computing time					For minimum value				
		PSO	**GA**	**OFA**	**CSO**	**SA**	**PSO**	**GA**	**OFA**	**CSO**	**SA**
Binomial estimator	Mean differences	−0.0099	−0.0731	−0.072	−0.1244	−1.1323	−330	−18	−160	0.026	2300
	Confidence interval	[−0.0976, −0.0043]	[−0.1238, −0.0223]	[−0.1208, −0.0137]	[−0.3305, −0.0321]	[−2.6860, −0.5927]	[−7.00e10, −10.00]	[−1.00e10, 40.00]	[−2.40e11, 0.052]	[−3.70e11, 70.00]	[0.0180, 2.66el5]
HL estimator	Mean differences	−0.01905	−0.0733	−0.07165	−0.1658	−1.4183	−4.22e7	−1.15e6	−5.99e7	−9.785e7	7.05e8
	Confidence interval	[−0.0643, −0.0072]	[−0.1238, −0.0379]	[−0.1021, −0.0455]	[−0.2283, −0.0824]	[−2.7678, −0.8625]	[−7.6el5, −10.00]	[−1.5e11, 25.00]	[−1.5e15, −2.489]	[−3el5, 25.00]	[430, 5.16e16]
W		−448	−440	−426	−466	−528	−159	−93.0	−153	−79.0	205
Mean		0	0	0	0	0	0	0	0	0	0
SD		106.960	106.960	102.060	106.960	106.960	74.330	74.330	74.330	74.330	74.330
Z		4.1839	4.1092	4.1692	4.3522	4.9318	2.1324	1.2444	2.0517	1.0561	2.7512
p value		**1.433e-5**	**1.985e-5**	**1.529e-5**	**6.739e-6**	**4.073e-7**	**0.0165**	**0.1067**	**0.0201**	**0.1455**	**0.0030**

Table 5.
Comparison of Wilcoxon rank sum test by PSO_4 and other models.

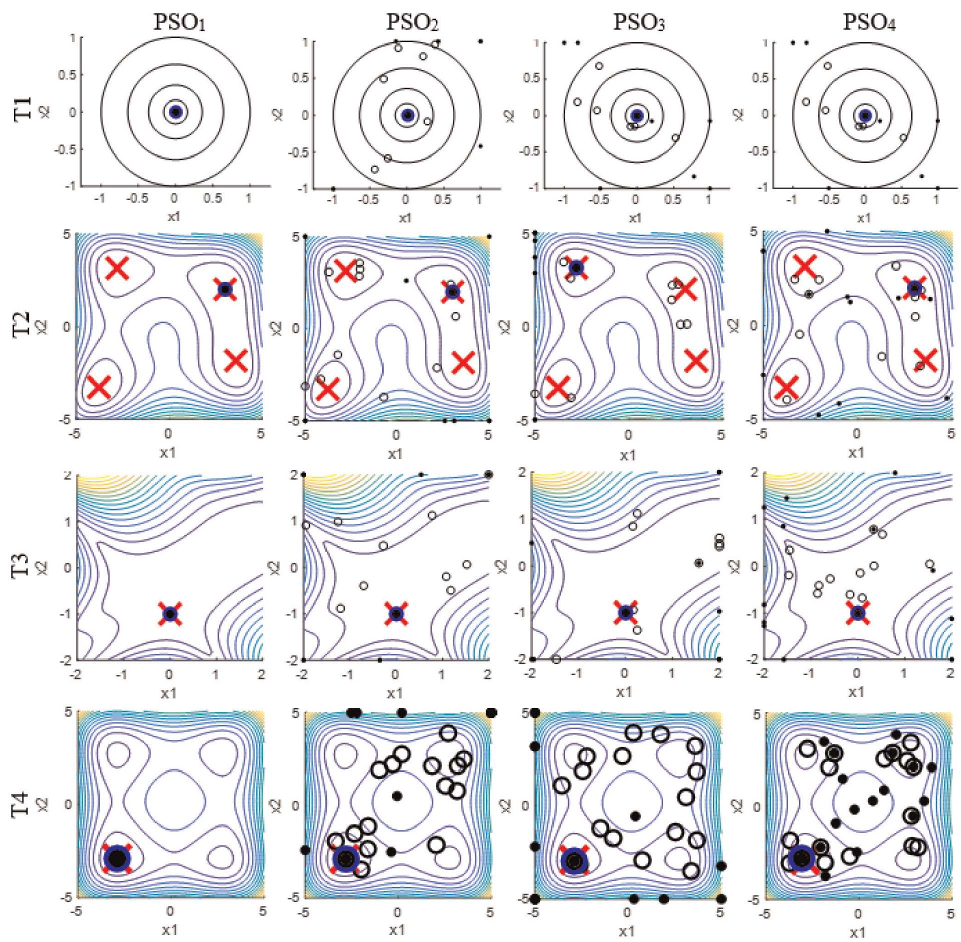

Figure 6.
Nonlinear optimization results by different test problems and PSO models.

These results show that there exists relative bigger difference between PSO_4 and PSO when calculating the running time, as well as the differences between PSO_4 and SA, while PSO_4 has low similarity in running speed performance and high similarity with these five typical algorithms in minimum value calculation and optimization observed by the indicators of mean differences and confidence interval. As for W, mean, SD and Z, all these models are alike. The p values for PSO, GA, OFA, CSO, and SA are all smaller than 0.05 in computing time, and they are also smaller than 0.05 for PSO, OFA, and SA, which manifests that the PSO_4 has superior performance in calculating speed or running time. Additionally, the new PSO_4 behaves more stable in solving CEC benchmark problems and has competitive performance in minimum value calculation and optimization.

6. Conclusion

This paper mainly proposes and investigates a nonlinear optimization method based on the improved particle swarm optimization algorithm. Different with the traditional

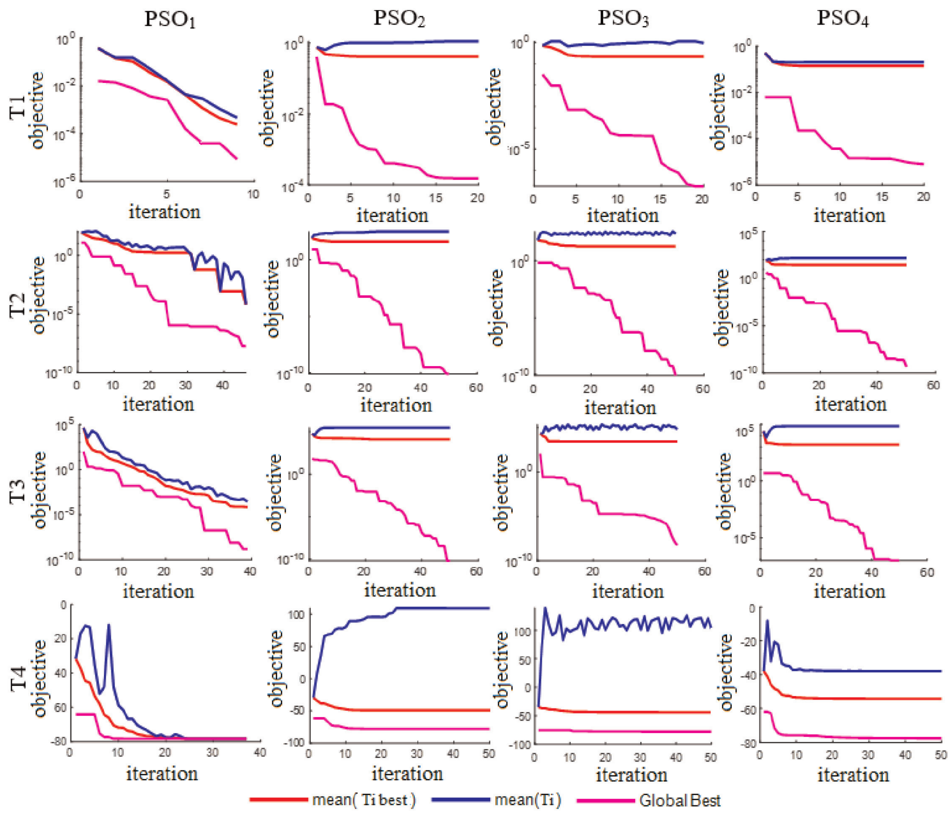

Figure 7.
Convergence curve for T1 to T4 test problems by PSO_1 to PSO_4 methods.

PSO method using a random searching pattern, the model and principles of the new PSO algorithms applying EM solving method are first introduced from the perspective of discretized solving process for SDE. Then, the influence of model parameters and input dimension on the nonlinear optimization characterizations are further revealed, and the feasibility of the proposed method is also validated regarding different nonlinear problems. Combining the optimization results of PFs and comparison with other nonlinear optimization methods, the capability and adaptability in solving various nonlinear problems are demonstrated and highlighted. Finally, the performance evaluation results manifests that the proposed method features faster convergence speed, accurate optimization results, as well as reliable and stable evolution process compared with other related optimization methods by CEC benchmark problem testing.

The future work may lie in solving high dimensional and more sophisticated nonlinear optimization problems with the adaptive parameter adjusting mechanism. Moreover, the promotion of the proposed nonlinear optimization method to various practical application is also of vital significance.

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Declaration of competing interest

The authors declare that they have no known competing financial interests that could have appeared to influence the work reported in this paper.

Data availability

The data used for the research will be made available on reasonable request.

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