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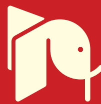
Teaching and Learning

Edited by Murat Tezer



Mathematics - Teaching and Learning

Edited by Murat Tezer

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Aims and Scope of the Series

Education and Human Development is an interdisciplinary research area that aims to shed light on topics related to both learning and development. This Series is intended for researchers, practitioners, and students who are interested in understanding more about these fields and their applications.

Meet the Series Editor

Katherine Meltzoff received her BA in Psychology from Trinity College, in Connecticut, USA and her Ph.D. in Experimental Psychology from the University of California, San Diego. She completed her postdoctoral work at the Yale Child Study Center with Dr. James McPartland. Dr. Meltzoff' doctoral dissertation explored neural correlates of reward anticipation to social versus nonsocial stimuli in children with and without autism spectrum disorders (ASD). She has been a faculty member at the University of California, Riverside in the School of Education since 2016. Her research focuses on translational studies to explore the reward system in ASD, as well as how anxiety contributes to social challenges in ASD. She also investigates how behavioral interventions affect neural activity, behavior, and school performance in children with ASD. She is also involved in the diagnosis of children with ASD and is a licensed clinical psychologist in California. She is the Assistant Director of the SEARCH Center at UCR and is a faculty member in the Graduate Program in Neuroscience.

Meet the Volume Editor

Murat Tezer was born in Nicosia in 1972. He earned a Ph.D. degree in Applied Mathematics and Computer Science at Eastern Mediterranean University in 2003. He worked as a project advisor at numerous schools under the European Union-supported grant program. More than 100 research publications and books/chapters on mathematics education have been published and cited in numerous international publications by prominent publishing houses. He directed more than 40 master's/doctoral theses as an advisor. He worked as the vice head of Computer Education and Instructional Technology. In 2008, he became the head of the Department of Elementary and Secondary Mathematics Education, a position he held from 2009 to 2018. He worked as Vice-Dean of the Education Faculty of Near East University between 2013 and 2018, and then he became a professor of mathematics education. Additionally, he serves as an executive board member of the Cyprus Educational Sciences Association.

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Preface

Mathematics is more than numbers and symbols; it is a universal language that transcends cultural and disciplinary boundaries, offering tools for analyzing complexity, structure, and the fundamental relationships that shape our world. As education adapts to 21st-century demands, mathematics teaching and learning have become a central concern, influenced by emotional, cognitive, technological, and philosophical factors.

This book approaches mathematics as both an academic subject and a practical skill. It examines how students learn mathematics, the challenges they encounter, and the teaching methods that are most effective. By combining educational theory, psychology, and technology, it provides a comprehensive view of mathematics education that acknowledges both its complexity and flexibility.

Mathematics education evolves in response to societal needs, classroom dynamics, and global discourse. Learners are no longer passive recipients but active participants who engage with abstract concepts, interpret real-world data, and develop strategies for novel challenges. In this way, learning mathematics becomes a process of cognitive growth, self-discovery, and creative problem-solving.

The following chapters provide an in-depth examination of this process. They explore how inquiry-based learning enhances mathematical understanding, how emotions influence engagement, how precision aids clarity, and how educators can connect complexity with comprehension. Discussions address student agency, reflective teaching, digital tools, and the teacher's evolving role—not as a mere disseminator of knowledge, but as a guide for meaningful learning.

For researchers, educators, policymakers, and students alike, this volume contributes to the ongoing dialogue about mathematics education. It does not propose definitive solutions but presents thoughtful questions, diverse perspectives, and practical strategies. Our aim is to encourage reflection, experimentation, and innovation—making this book both a resource and a catalyst for more inclusive, meaningful, and effective mathematics education.

Chapter 1 shows how inquiry-based learning helps math students. When students explore concepts through Inquiry-Based Learning, they engage more deeply, think critically, and achieve better results. This approach turns learners into discoverers, and teachers are encouraged to guide rather than lecture.

Chapter 2 examines math anxiety through research, investigating how teaching approaches, teacher support, and classroom environments impact students' confidence and resilience. It provides guidance on creating supportive learning environments.

Chapter 3 explains how precise definitions strengthen mathematical theory and help students grasp abstract concepts more effectively. It highlights how language shapes understanding in mathematics.

Chapter 4 combines the rigidity of complex systems theories with the interpretive flexibility of soft approaches. It introduces a new approach to examining complexity in mathematics education through the concepts of “substructure and superstructure”, moving beyond traditional divisions to enhance curriculum design.

Chapter 5 traces the historical and pedagogical evolution of mathematics in the physical sciences. It identifies cognitive and conceptual hurdles faced by first-year STEM students and suggests pedagogical interventions that bridge the gap between mathematical abstraction and empirical reasoning.

Chapter 6 examines how new mathematics teachers use digital visualization tools in under-resourced schools. It explores both the potential benefits and practical challenges of technology integration, showing how educators adapt creatively to limited resources.

Chapter 7 examines how cloud computing revolutionizes mathematics laboratories. Cloud-based platforms enhance computational pedagogy by decentralizing access, facilitating collaborative engagement, and improving numerical experimentation, feedback loops, and learner autonomy.

Chapter 8 explores graph theory games, examining how game-based learning develops strategic thinking, mathematical modeling, and motivation. It advocates integrating mathematical games into the curriculum to enhance engagement and conceptual mastery.

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Section 1

Rethinking Pedagogy: Cognitive and Emotional Foundations in Mathematics Learning

Chapter 1

Empowering Learning: The Role of Inquiry-Based Learning in Enhancing Mathematics Achievement

Elaina Khasawneh

Abstract

This chapter examines the implementation of inquiry-based learning (IBL) in college mathematics and its impact on student achievement, particularly in problem-solving, conceptual understanding, and mathematical reasoning. Grounded in a constructivist framework, IBL engages students through structured inquiry activities, collaborative learning, and formative assessments, fostering deeper comprehension of mathematical concepts. The study addressed in this chapter found that students in an IBL setting demonstrated stronger reasoning skills, enhanced problem-solving abilities, and deeper mathematical comprehension compared to those in traditional lecture-based classrooms. The study also noted improvements in retention and engagement, though the primary focus was on student achievement. The chapter also discusses key challenges in implementing IBL, including student resistance, instructor facilitation, and balancing inquiry with curriculum requirements. By integrating empirical evidence with instructional strategies, this chapter provides educators with practical insights for effectively incorporating inquiry-driven learning in mathematics instruction.

Keywords: inquiry-based learning, constructivism, active learning, student engagement, guided inquiry, collaborative learning, problem-solving skills, student-centered learning, formative assessment, instructional strategies, facilitator role, probing questions

1. Introduction

Mathematics education is at a crossroads. While traditional lecture-based methods have long been the standard, the rapidly changing demands of the workforce and the increasing diversity of learners necessitate innovative instructional strategies. Inquiry-based learning (IBL) has emerged as a transformative approach that not only fosters mathematical understanding but also develops critical thinking and problem-solving skills.

As the economy shifts from an industrial model to an information-driven one, the skills required for success have also evolved. Higher education institutions must

adapt their curricula to align with these changes, ensuring that college graduates are prepared for the modern workforce [1]. Educators play a key role in this transformation by shifting their teaching approaches from exam-focused instruction to fostering lifelong learning.

Today's workforce requires professionals with strong critical thinking, collaboration, and problem-solving skills [2, 3]. In fields like technology, analytical reasoning is valued over rote memorization, and professionals must apply knowledge dynamically rather than relying solely on textbook learning. Given this shift, higher education should adopt instructional models that encourage independent learning and cultivate critical thinking [3–5].

Traditional lecture-based teaching methods, designed for the industrial era, were created to efficiently deliver content. However, these approaches emphasize memorization and standardized content rather than deeper understanding and skill application. As a result, students often struggle to engage with the material in meaningful ways, limiting their ability to think critically, solve problems flexibly, and apply knowledge beyond the classroom [5–9].

Students who engage in active learning retain significantly more information than those who rely solely on lectures [6]. Research on constructivist learning shows that exploring and discussing mathematical problems leads to a deeper understanding compared to passive information reception [10]. These findings emphasize the need for learning environments that promote active engagement with mathematical concepts.

One area where the need for instructional change is particularly evident is college algebra, a course that serves as a major roadblock for many students pursuing degrees in STEM and other fields [11]. College algebra is a critical course designed to develop students' problem-solving, critical thinking, and communication skills—essential for both academic and professional success. However, this course also has one of the highest failure rates nationwide. Recent studies indicate that approximately 50% of the one million students who enroll in college algebra annually fail to earn a grade of "C" or better [12]. Many students are required to retake the course multiple times, while others drop out of college entirely due to the algebra requirement [11].

The Conference Board of the Mathematical Sciences [13] found that traditional instructional methods dominate college algebra courses, with 79% of community college classes and 66% of four-year institution classes emphasizing memorization and procedural learning [14]. Students who struggle in algebra often experience the same ineffective instructional methods repeatedly, with little change in teaching approaches to support their learning needs [15].

Given algebra's critical role in higher education and employment, the persistence of poor performance in math underscores the urgent need for a shift toward constructivist teaching approaches. Inquiry-based learning (IBL) offers an alternative that may address these long-standing challenges, providing students with opportunities to engage actively with mathematical concepts rather than passively receiving information [9, 16].

1.1 The need for innovative teaching methods

Lecture-based instruction has long been the dominant method in mathematics classrooms. While this approach efficiently delivers content, it often fails to engage students in meaningful problem-solving and conceptual understanding. Inquiry-Based Learning (IBL) offers an alternative that actively involves students in mathematical exploration, collaboration, and reasoning.

For instance, instead of simply lecturing on quadratic functions, an IBL classroom might have students design a real-world model—such as predicting revenue growth in a small business—where they must apply algebraic reasoning collaboratively. This kind of application highlights the practicality of IBL and allows students to see how mathematical concepts function beyond the classroom.

Research further supports the effectiveness of IBL in college mathematics. Kogan and Laursen [17] conducted a multi-institutional study examining student outcomes from IBL across various college mathematics courses. Their findings showed that students in IBL-based courses demonstrated significantly greater learning gains compared to those in traditional lecture-based courses. Specifically, students in IBL settings exhibited improved problem-solving skills, deeper conceptual understanding, and more positive attitudes toward mathematics. These results reinforce the argument that shifting toward IBL can enhance student performance in college algebra and better prepare students for success in mathematical problem-solving beyond the classroom [18].

These findings also highlight practical challenges and instructional strategies for integrating IBL into college algebra. As discussed further in later sections, understanding these challenges provides valuable insights for educators looking to refine their teaching approaches and improve student learning experiences. Addressing these aspects can help faculty implement more effective, inquiry-driven teaching practices, ultimately preparing students to apply algebraic concepts with confidence beyond the classroom.

The success of IBL in mathematics education is not incidental—it is deeply rooted in well-established theories of learning. Constructivism, as developed by scholars like Dewey, Piaget, and Vygotsky, explains why students learn best when they engage in inquiry, collaboration, and hands-on problem-solving. Understanding these theoretical foundations helps educators implement IBL more effectively and create learning environments that foster deeper engagement and mathematical understanding.

1.2 Purpose and scope of the chapter

This chapter explores how Inquiry-based learning (IBL) enhances mathematics achievement by increasing student engagement, fostering collaboration, and promoting deeper understanding. It examines the theoretical foundations of IBL, presents empirical evidence on its effectiveness, and offers practical strategies for implementation. Additionally, the chapter addresses challenges that come with adopting IBL, such as student resistance, the role of the instructor as a facilitator, and balancing inquiry with curriculum coverage.

By integrating research findings with instructional strategies, this chapter provides educators with the knowledge and tools needed to implement IBL effectively in college mathematics courses. The goal is to highlight how active, inquiry-driven learning environments can improve student engagement, retention, and overall mathematical achievement.

2. Theoretical foundations of inquiry-based learning (IBL)

Inquiry-based learning (IBL) is deeply grounded in well-established learning theories that emphasize active engagement, student-centered pedagogy, and inquiry-driven exploration. Traditional lecture-based instruction often focuses on memorization and procedural knowledge, while IBL shifts the emphasis to deeper conceptual understanding, critical thinking, and problem-solving. This section explores the theoretical

underpinnings of IBL, highlighting the contributions of constructivist theorists, the role of active learning, and the impact of inquiry in mathematics education.

2.1 Constructivist learning theory (Piaget, Vygotsky, Dewey)

Constructivist learning theory serves as the foundation for IBL, asserting that students actively construct knowledge rather than passively receiving information [19–21]. Each contributed key insights into how students develop understanding through experience, social interaction, and inquiry-driven exploration.

- Learning as an active process—students must engage in inquiry-based tasks, constructing their own understanding rather than memorizing procedures [21].

- Social construction of knowledge—learning takes place in a social context, where peer collaboration and instructor guidance shape cognitive development [20, 21].

- Connecting prior knowledge to new experiences—emphasized that learning happens when students connect new information to their existing knowledge, allowing for deeper comprehension [19].

- The role of the teacher as a facilitator—instructors guide discussions, structure collaborative learning, and encourage students to explore concepts independently [19, 21]

- Lifelong learning and critical thinking—learning is a continuous process that extends beyond the classroom, reinforcing problem-solving skills that remain relevant throughout life [19, 20].

By applying these principles, IBL fosters an environment where students take ownership of their learning, ask meaningful questions, and seek solutions through exploration and collaboration. This shift from passive absorption to active engagement enables students to build a deeper conceptual understanding of mathematics.

2.2 Active learning and student-centered pedagogy

IBL aligns with student-centered learning, shifting the focus from the instructor's delivery of information to the student's active engagement in knowledge construction. Unlike traditional lecture-based instruction, which often prioritizes content delivery, IBL immerses students in hands-on exploration, collaborative discussions, and real-world problem-solving.

An IBL classroom is structured around four key characteristics that promote engagement and deeper comprehension:

- Learner-centered approach: instruction builds upon students' prior knowledge, allowing them to develop understanding through exploration and discovery [22, 23].

- Knowledge-centered learning: the focus shifts from rote memorization to conceptual mastery, encouraging students to develop reasoning skills and apply their knowledge in diverse contexts [24].

- Assessment-centered instruction: formative assessments, including reflective discussions, student-led problem-solving, and instructor feedback, guide learning and help tailor instruction to meet student needs [25].

- Community-centered engagement: learning becomes a collaborative process, where students analyze, critique, and refine their understanding through structured inquiry and group discussions [26].

Dewey [27] was critical of traditional teaching models that prioritized memorization over critical thinking, arguing that meaningful education must involve active engagement and inquiry. Similarly, Bruner [28] highlighted the importance of connecting instruction to students' real-world experiences to make learning more meaningful and relevant.

By embedding these principles into instruction, IBL creates dynamic, student-centered learning environments that improve mathematical comprehension while equipping students with essential critical thinking and problem-solving skills.

2.3 The role of inquiry in deepening mathematical understanding

IBL fosters deeper mathematical understanding by engaging students in exploration, reasoning, and problem-solving rather than passive content absorption. When students actively analyze patterns, test hypotheses, and construct their own explanations, they develop a stronger conceptual foundation that enhances both retention and application of mathematical knowledge.

The essence of IBL lies in learning through inquiry, where students engage in structured problem-solving instead of simply memorizing formulas. This approach allows students to:

- Discover mathematical relationships and apply their knowledge in meaningful contexts.

- Justify their reasoning and refine their thought processes through structured inquiry.

- Move beyond procedural fluency to develop a conceptual grasp of mathematical principles.

Kogan & Laursen [17] found that students in inquiry-based mathematics courses demonstrated improved reasoning abilities and a greater willingness to engage with complex problems. Similarly, Magee and Flessner [29] observed that inquiry-based instruction fosters a culture of exploration and intellectual independence, which translates to higher achievement in mathematics.

Instructors play a crucial role in facilitating inquiry by posing probing questions, encouraging discussion, and guiding students through the problem-solving process. Instead of delivering answers, they create opportunities for students to engage dynamically with mathematical ideas.

By fostering ownership of learning, IBL encourages students to approach mathematics with curiosity and confidence. This transformation shifts mathematics education from a focus on rote memorization to an emphasis on conceptual mastery and real-world application.

3. Inquiry-based learning in mathematics education

To better understand IBL's role in mathematics education, it is important to examine its defining characteristics and instructional approach, as these elements shape how students engage with mathematical concepts and develop problem-solving skills.

3.1 Definition and key characteristics of IBL

Inquiry-based learning (IBL) is an instructional approach that actively engages students in the learning process by encouraging them to investigate mathematical concepts through questioning, problem-solving, and collaboration. IBL places students at the center of their learning experience, allowing them to build knowledge through active inquiry rather than passively receiving information. This shift fosters curiosity, critical thinking, and a deeper understanding of mathematical principles [30].

At the core of IBL is the belief that learning happens best when students take an active role in their education. Through guided investigation, students work through mathematical problems, test ideas, and refine their reasoning. This approach not only strengthens their conceptual understanding but also enhances their ability to think independently and apply mathematical knowledge in various contexts [29, 30].

Several key characteristics define IBL in mathematics education:

- Student-centered learning: students take ownership of their learning by engaging with mathematical concepts through inquiry and discovery [22, 23].

- Problem-solving through inquiry: instead of focusing solely on procedures, students work through mathematical problems by analyzing patterns, making connections, and developing their own solutions [24].

- Collaborative learning: Peer interaction plays a vital role in the inquiry process, fostering discussion, critical thinking, and shared problem-solving [26].

- Instructor as facilitator: rather than providing direct instruction, educators guide students through open-ended discussions, prompting them to explore different strategies and deepen their understanding [19, 31].

- Real-world applications: mathematical concepts are introduced in meaningful contexts, helping students connect abstract ideas to practical situations [32, 33].

- Continuous feedback and reflection: learning is reinforced through formative assessments, where students reflect on their progress, receive feedback, and refine their problem-solving strategies [34].

By integrating these elements, IBL transforms the learning experience into a dynamic and interactive process. It encourages students to question, analyze, and engage deeply with mathematical ideas, equipping them with the skills needed to approach complex problems with confidence [35].

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3.2 IBL versus traditional learning

A key distinction between IBL and traditional lecture-based instruction lies in the role of the student and instructor. In a conventional lecture setting, the teacher delivers content, and students passively absorb information through notetaking and memorization. In contrast, IBL fosters active participation, where students generate their own understanding by engaging with problems, collaborating with peers, and refining their reasoning (**Table 1**).

Studies indicate that students in IBL classrooms exhibit higher retention rates and deeper conceptual understanding than those in traditional settings. By shifting the focus from content delivery to active inquiry, IBL enables students to develop essential problem-solving skills and mathematical reasoning abilities.

3.3 Benefits of IBL in mathematics learning

Inquiry-based learning (IBL) offers several advantages in mathematics education, particularly in fostering deeper understanding and improving student outcomes. By shifting the focus from passive content delivery to active student engagement, IBL enhances the learning experience and helps students develop essential mathematical skills.

Some of the key benefits of IBL in mathematics include the following:

- Stronger problem-solving skills: IBL encourages students to think critically, analyze different approaches, and develop reasoning skills rather than relying on memorized formulas [29].

- Higher student engagement: the interactive nature of IBL makes learning mathematics more engaging and reduces the anxiety often associated with the subject [36].

- Deeper conceptual understanding: students gain a stronger grasp of mathematical concepts by working through problems, discussing ideas, and making connections rather than passively receiving information [33].

Feature	Traditional learning	Inquiry-based learning (IBL)
Teaching approach	Teacher-centered, lecture-based	Student-centered, problem-solving-based
Role of the teacher	Authority, knowledge transmitter	Facilitator, inquiry guide
Role of the student	Passive listener, note-taker	Active participant, problem solver
Assessment focus	Memorization, standardized tests	Formative, continuous feedback
Collaboration	Limited or individual work	Group-based, interactive learning
Feedback mechanisms	Summative assessments and periodic exams	Ongoing formative assessments with reflective feedback
Learning environment	Rigid, structured classroom	Flexible, dynamic, and collaborative learning space
Application of knowledge	Focus on abstract, theoretical concepts	Focus on real-world applications and practical problem solving

Table 1.
IBL versus traditional learning.

- Encouragement of independent learning: IBL fosters curiosity and self-directed learning, preparing students to apply mathematical reasoning beyond the classroom [24].

- Improved knowledge retention: research suggests that students retain mathematical concepts more effectively when they discover patterns and relationships through inquiry rather than memorization [22].

- Stronger connections to real-world applications: IBL allows students to see the relevance of mathematics by applying concepts to real-world situations, making the subject more meaningful and applicable to everyday life [33].

Research by Kogan and Laursen [17] shows that IBL boosts students' confidence in problem-solving and strengthens their mathematical reasoning. Similarly, Freeman et al. [37] found that active learning approaches like IBL enhance academic performance in STEM disciplines. By integrating IBL, instructors can foster student ownership of learning, deeper engagement with mathematical concepts, and critical thinking skills. The next section outlines practical strategies for implementing IBL effectively in college mathematics classrooms.

4. Implementing IBL in college mathematics

Implementing inquiry-based learning (IBL) in college mathematics requires a well-structured approach to ensure student engagement and meaningful learning experiences. Lessons should be designed to encourage students to explore mathematical concepts, pose questions, and collaborate with peers.

Strategies for structuring inquiry-based lessons include the following:

- Designing meaningful inquiry activities: IBL begins with open-ended tasks that encourage students to engage with mathematical concepts beyond procedural understanding [38]. Activities should promote critical thinking by requiring students to analyze, synthesize, and evaluate mathematical problems rather than simply recalling formulas.

- Facilitating a collaborative learning environment: collaboration is central to IBL, as students construct knowledge through group discussions and shared problem-solving [29, 39]. Instructors should create a classroom culture that encourages intellectual risk-taking, allowing students to develop deeper reasoning skills and confidence in their abilities [40].

- Integrating technology to support inquiry: digital tools enhance IBL by providing interactive learning experiences, real-time simulations, and opportunities for collaborative problem-solving [31, 41]. Studies suggest that technology integration within IBL classrooms increases student engagement and supports deeper conceptual understanding [41, 42].

- Implementing formative and summative assessments: formative assessments, such as reflective journals, peer discussions, and instructor feedback, help monitor student thinking and adjust instruction as needed [43]. Summative

assessments, including presentations and written reports, encourage students to articulate their reasoning and demonstrate their inquiry-driven learning [43].

- Encouraging reflective thinking: Reflection is a key component of IBL, allowing students to assess their understanding, refine problem-solving strategies, and develop a stronger grasp of mathematical concepts [44].

By implementing these strategies, educators can foster a dynamic and engaging learning environment that promotes critical thinking, problem-solving, and collaboration. The next section examines how these strategies have been successfully applied in college algebra classrooms and their impact on student learning outcomes.

5. Empirical evidence: Impact of IBL on mathematics achievement

Understanding the effects of Inquiry-Based Learning (IBL) on student achievement requires empirical analysis of its implementation in college algebra. This section presents findings from a study conducted at a mid-sized university, examining how IBL influences student engagement, conceptual understanding, and academic performance compared to traditional lecture-based instruction.

The following subsections outline the study's research objectives, design, methodology, and key findings, providing insights into the role of IBL in improving mathematics achievement and student learning experiences.

5.1 Introduction to the study

This study explores the impact of IBL in a college algebra course at a mid-sized university, focusing on student engagement, conceptual understanding, and academic achievement compared to traditional lecture-based instruction. While previous studies highlight the benefits of active learning in STEM education, limited research specifically examines IBL's role in foundational mathematics courses. Given that college algebra serves as a gateway course for STEM and non-STEM majors, understanding its impact is critical for improving retention and success rates.

5.2 Research objectives

The study in this chapter focused on the impact of inquiry-based learning (IBL) on student achievement in a college algebra classroom, with student engagement and perceptions considered as secondary factors. The primary goal was to examine how IBL influenced learning outcomes and academic performance. Specifically, the research explored whether an inquiry-based learning intervention led to measurable improvements in students' mathematics achievement scores.

In addition to assessing achievement, the study also examined how IBL affected student engagement and participation in the classroom, as well as how students perceived the effectiveness of IBL as an instructional approach. While these aspects provided valuable context, the main emphasis remained on understanding the extent to which IBL contributed to student success in mathematics. By addressing these questions, the study offered insight into the role of IBL in improving learning outcomes while also considering its impact on classroom engagement and student experience.

5.3 Research design and sample selection

This study employed a quasi-experimental, pre-test/post-test control group design. A quantitative approach was used to measure student learning gains, while qualitative observations and self-report surveys provided additional insight into student engagement and instructional effectiveness. The IBL group participated in student-centered, inquiry-driven instruction, engaging in collaborative problem-solving and guided exploration, while the TL group followed a traditional, instructor-led format, where learning was primarily lecture-based, and students were expected to absorb information passively. The pre-test was administered at the beginning of the semester, and a post-test at the end, allowing for a comparative analysis of learning gains.

Participants were selected from a college algebra course at a Predominantly Black Institution (PBI) in the United States. This course was chosen due to its role as a required class for various degree programs, making it an ideal setting for examining the impact of different instructional methods. A total of 41 students participated in the study, with 23 students assigned to the IBL group and 18 students assigned to the TL group. Students were enrolled in their respective sections based on the institution's standard registration process rather than randomized assignment. Importantly, students were not informed that the sections differed in pedagogical approach; they registered for their courses without prior knowledge of the instructional methods that would be used. While the lack of randomization presents limitations, the use of ANCOVA helped control for pre-existing differences in students' mathematical readiness.

5.4 Research design and methodology

To evaluate the effectiveness of Inquiry-Based Learning (IBL) in college algebra, this study employed a structured research design comparing IBL with traditional lecture-based instruction. The study aimed to assess how IBL influenced student engagement, mathematical achievement, and perceptions of instructional effectiveness. By examining two course sections—one utilizing IBL and the other following a traditional lecture format—the research provided empirical insights into the impact of active learning strategies in mathematics education. The methodology included participant selection, instructional implementation, and data collection through pre- and post-tests, observations, and student surveys, ensuring a comprehensive assessment of IBL's effectiveness.

5.5 Data collection and measures

To assess student achievement, a college algebra readiness pre-test was administered at the start of the semester to establish baseline knowledge. A post-test was administered at the end to measure conceptual growth and problem-solving improvement. ANCOVA (analysis of covariance) was used to adjust for initial differences in pre-test scores, ensuring that post-test differences reflected instructional impact rather than prior ability.

In addition to achievement data, classroom engagement was assessed using instructor observations, including the types of questions students asked (clarifying vs. higher-order thinking), peer-to-peer collaboration in problem-solving activities, and consistency of attendance and participation. Students also completed anonymous

self-report surveys at the end of the course, measuring their confidence in problem-solving skills, perceptions of instructional effectiveness, and engagement levels. These survey responses provided qualitative insights into how students perceived the effectiveness of IBL versus traditional lecture-based instruction.

5.6 Implementing IBL strategies: Examples from my classroom

IBL was incorporated into the college algebra curriculum through structured activities designed to promote student inquiry and collaboration. The approach started with guided support and gradually shifted toward student-led learning. In this section, I will outline the key strategies used and explain how each was implemented in the classroom.

5.6.1 Designing meaningful inquiry activities

When introducing quadratic functions, instead of immediately giving students the standard equation, I present a real-world scenario: "*Imagine you toss a ball straight up into the air. It rises, slows down, reaches a peak, and then falls back to the ground. If we recorded the height of the ball at different times, how could we mathematically model its motion?*"

- Students start by analyzing a table of values representing the height of the ball at different time intervals.
- I guide the discussion with questions like, "*What patterns do you see?*" and "*Where do you think the highest point occurs?*".
- Rather than directly teaching the quadratic formula, I let them use their observations to predict how the motion might be represented mathematically.
- As they test different possibilities, they naturally recognize that the motion follows a curved path, leading them to the idea of a quadratic equation.

This approach allows students to discover the mathematical relationship on their own, making the concept more intuitive and engaging before introducing procedural steps.

As students engaged with the problem, instructor support gradually decreased, encouraging independent reasoning and problem-solving. This approach, known as scaffolding, ensures that students develop confidence and autonomy in their learning process. Recent research supports this method, with Kumazah et al. [45] highlighting that inquiry-based scaffolding through guided questioning prepares students for deeper mathematical exploration, fostering both understanding and independence.

This structured inquiry approach seamlessly transitions into the next phase of learning—collaborative engagement. By first exploring concepts independently and developing initial reasoning skills, students enter group discussions better prepared to test models, challenge reasoning, and refine their equations. This progression from individual inquiry to peer collaboration mirrors the findings of Siller and Ahmad [46], who emphasize that when students engage in inquiry-based exploration before collaborative discussions, they exhibit higher engagement and deeper conceptual learning.

5.6.2 Facilitating a collaborative learning environment

Collaboration was essential to the IBL implementation, allowing students to test models, challenge reasoning, and refine their equations. Initially, structured roles such as data recorder, graphing tool user, and discussion leader ensured active participation, particularly helping students hesitant to engage. Research supports this approach, showing that structured collaboration enhances mathematical understanding and engagement [47, 48].

As students gained confidence, collaboration became more flexible, with discussions developing naturally. They began questioning each other's work, debating strategies, and constructing arguments, reflecting findings by Gillies [49] that peer discussions improve mathematical reasoning. One student who initially struggled with identifying key points in a quadratic function developed a clearer understanding through peer discussions, reinforcing Siller and Ahmad's [46] research on collaboration improving problem-solving skills and attitudes toward mathematics.

A key enhancement in my classroom was integrating Desmos, replacing earlier reliance on TI-84 calculators. Desmos allowed students to manipulate equations, visualize transformations, and compare quadratic functions in real time, making abstract concepts more accessible.

Through structured guidance and increasing independence, peer collaboration, supported by technology, helped shift students from passive learning to active inquiry. As students engaged in IBL, effective assessments became critical in evaluating and guiding their learning. The next section explores how formative and summative assessments were integrated into the IBL framework to support student achievement.

5.6.3 Integration of formative and summative assessments

Assessing student learning in IBL requires a balance of formative and summative assessments to provide both immediate feedback and a measure of overall mastery. Formative assessments, such as reflections, peer discussions, and instructor feedback, help students refine their understanding throughout the learning process, while summative assessments evaluate conceptual mastery at the end of instructional units. Research emphasizes the importance of this dual approach, with Koksalan and Ogan-Bekiroglu [44] highlighting that formative assessments foster deeper conceptual understanding by promoting continuous feedback and self-reflection. Similarly, Chand and Pillay [50] found that blending formative and summative assessments enhances learning outcomes, ensuring that assessments serve as tools for growth rather than just measures of achievement.

Beyond structured assessments, student reflections played a crucial role in the learning process. Weekly reflections allowed students to analyze their progress, identify challenges, and assess their problem-solving strategies. Guided prompts encouraged them to think critically about their learning experience, compare IBL with traditional instruction, and connect mathematical concepts to real-world applications. Reviewing these reflections helped identify common struggles, provided opportunities for individualized feedback, and reinforced student ownership of learning.

This structured assessment approach not only measured student understanding but also shaped their learning experience in meaningful ways. By embedding

assessment within the learning process rather than treating it as a separate task, students developed a stronger conceptual foundation while refining their reasoning skills. The next section presents key findings from the study, illustrating how IBL impacted student engagement, performance, and confidence in college algebra.

5.6.4 Encouraging reflection for deeper understanding

Reflection is a crucial part of learning. When students take the time to think about their problem-solving process, they become more aware of their strengths and the areas where they need improvement. Encouraging them to write about their thought process, discuss their challenges, and explain their reasoning helps them develop a deeper understanding of mathematical concepts [44]. At the end of each week, I ask students to complete a brief reflection journal with prompts such as *What was the most challenging concept this week? How did you overcome a difficult problem? What connections can you make between this topic and real life?* One student once wrote, "*I used to just memorize formulas without really knowing why they worked. But when I started breaking problems down and thinking through each step, I realized I understood what I was doing. Now, I feel more confident tackling new problems.*"

These reflections help students internalize concepts, build confidence, and develop a habit of self-directed learning. The next section presents key findings from the study, illustrating how IBL impacted student engagement, performance, and confidence in college algebra.

6. Key findings and insights

6.1 Student performance and achievement

An ANCOVA analysis compared post-test performance between IBL and TL students, controlling for pre-test scores. The results confirmed a significant impact of instructional method on achievement ($F(1,38) = 20.70$, $p < .001$), with a large effect size ($\eta^2 = .35$), demonstrating that IBL significantly improves mathematics achievement over traditional lecture-based instruction.

IBL's structured inquiry process led students to engage more deeply with concepts rather than memorizing formulas. By attempting problems independently, discussing reasoning with peers, and receiving instructor guidance, IBL students demonstrated stronger problem-solving skills. One student noted, "My professor first let us try to solve the problem ourselves, then work with peers, and finally reviewed it as a class. I like this method because it makes the work less complicated for those who are not great at Algebra."

Pre- and post-test comparisons showed greater conceptual gains in the IBL group. Open-ended assessments further revealed that IBL students better articulated reasoning and applied algebraic principles to new problems, reinforcing both procedural fluency and conceptual mastery.

These findings align with prior research showing IBL enhances learning. Studies by Abdi [50] and Laursen et al. [18] found IBL students outperformed their traditionally taught peers in post-tests. IBL improved problem-solving and conceptual understanding in mathematics.

By shifting from passive learning to inquiry-driven engagement, IBL fosters deeper understanding and retention of mathematical concepts [51].

6.2 Student engagement and participation

Student engagement varied between the two groups, with IBL students showing more involvement in discussions, problem-solving, and peer collaboration. Unlike their peers in the traditional lecture-based (TL) section, students in the IBL section actively participated by asking questions, exchanging ideas, and working closely with both their classmates and the instructor. This shift from passive learning to active engagement not only improved their confidence but also enhanced their ability to approach problems independently. Observations indicated that students in the lecture-based section were more passive, often waiting for the instructor to provide solutions rather than actively working through problems. In contrast, IBL students frequently engaged in discussions, collaborated on complex problems, and took ownership of their learning.

To assess engagement, participation and attendance were recorded and analyzed throughout the semester. Classroom observations focused on student interactions, the depth of questioning, and contributions to discussions. The impact of IBL on retention was noticeable—92% of students in the IBL section remained enrolled throughout the semester, compared to only 32% in the TL section. Although retention was not the primary focus of this study, this pattern aligns with previous research suggesting that active learning strategies contribute to student persistence, particularly in STEM disciplines, where mathematics often serves as a significant barrier to degree completion. Theobald et al. [39] found that active learning reduces achievement gaps and increases retention for underrepresented students in STEM fields, further supporting the idea that engagement-driven approaches contribute to student success.

Existing research also highlights the benefits of IBL in promoting student engagement. Presnillo and Aliazas [52] found that IBL enhances cognitive, behavioral, and emotional engagement, as well as critical thinking skills. Similarly, Yu et al. [53] concluded that students in IBL classrooms demonstrate stronger reasoning abilities, higher engagement levels, and more effective peer collaboration. These findings align with the observations in this study, where IBL students engaged more deeply with the material, worked effectively in groups, and developed a stronger sense of ownership over their learning. In contrast, TL students often reported difficulties engaging with the material, citing a lack of interaction and limited opportunities for collaboration.

This research reinforces the argument that active learning strategies, such as IBL, not only foster engagement but also help students develop essential academic skills like problem-solving and critical thinking. By creating a learning environment that encourages participation, IBL proves to be an effective instructional approach for increasing both student engagement and retention in mathematics education [54–56].

6.3 Student perspective of the instructional pedagogy

Students in the IBL section viewed the instructional approach as a positive learning experience. Many expressed feeling more engaged and found that the collaborative, inquiry-driven environment helped them develop a deeper understanding of mathematical concepts. Unlike traditional lecture-based instruction, which often felt passive, IBL encouraged active participation, critical thinking, and a greater sense of ownership over their learning. These observations align with research by Laursen et al. [18], which found that IBL in college mathematics courses led to increased student participation and improved learning outcomes compared to traditional lecture-based instruction.

Students also reported that IBL boosted their confidence in solving problems independently. They felt more comfortable asking questions, discussing mathematical concepts with peers, and explaining their reasoning. Several students shared that the interactive nature of IBL made learning more engaging and less intimidating, easing the anxiety often associated with mathematics. These findings mirror the research by Smith and MacGregor [48] which showed that inquiry-based environments positively influence students' attitudes toward mathematics, fostering deeper engagement and stronger self-efficacy.

Observations further indicated that students in the IBL section were more likely to attend class regularly, collaborate with classmates, and engage with the material outside of class. Many appreciated the structured inquiry approach, which allowed them to explore different problem-solving strategies while still receiving guidance and feedback from the instructor. This balance of autonomy and support contributed to their overall satisfaction with the learning experience.

Student reflections reinforced these observations. One student noted, "*I liked that I got to try solving problems on my own first and then work in a group to compare answers. It helped me understand my own thinking while also learning from others.*" Another student shared, "*The instructor didn't just give us the answer—she gave us hints and let us figure it out. That really helped me learn how to approach problems instead of just memorizing steps.*" A third student explained, "*We had to discuss different ways to solve a problem, and that made me realize there isn't just one way to get the answer. It helped me understand the bigger picture.*"

Overall, students saw IBL as an effective instructional method that not only improved their mathematical understanding but also strengthened their problem-solving, communication, and critical thinking skills. These findings align with prior research advocating active learning, demonstrating that IBL fosters deeper comprehension, long-term retention of mathematical concepts, and greater student engagement [18].

7. Discussion and implications

These findings align with existing research on active learning and IBL, reinforcing the idea that inquiry-based approaches significantly enhance student engagement, conceptual understanding, and academic achievement in mathematics. The results of this study indicate that inquiry-based learning (IBL) is an effective instructional approach for improving student achievement in college algebra. The significant improvement in post-test scores among IBL students, compared to those in the traditional lecture-based (TL) section, suggests that active engagement and inquiry-driven learning foster deeper conceptual understanding.

Beyond test performance, IBL students demonstrated stronger problem-solving abilities, as reflected in their capacity to analyze complex problems, engage in collaborative discussions, and articulate mathematical reasoning. These results align with constructivist learning theory, which emphasizes that students construct knowledge actively through inquiry, exploration, and peer interaction rather than passively receiving information. By engaging with mathematical concepts through inquiry, students develop a more intuitive grasp of algebraic principles, enabling them to apply their knowledge more effectively in diverse contexts.

The findings from this study are consistent with broader research on IBL in mathematics instruction, which has demonstrated its effectiveness in enhancing student

engagement, critical thinking, and problem-solving skills. Studies by Hmelo-Silver [57] and Jonassen [58] highlight how inquiry-based learning strengthens students' problem-solving abilities and fosters critical thinking, both of which are essential for success in mathematics. Similarly, Smith and MacGregor [48] found that students in inquiry-driven classrooms develop a more meaningful and lasting understanding of mathematical concepts compared to those in traditional lecture settings.

Recent research further reinforces these findings. Presnillo and Aliazas [52] found that IBL significantly improves student engagement, motivation, and retention. Similarly, Calder [59] argues that shifting from traditional lecture-based instruction to student-centered, inquiry-driven approaches enhances both engagement and academic performance in mathematics. These studies align with the results of this research, further validating the effectiveness of IBL as a powerful instructional strategy for college algebra.

8. Challenges and recommendations for IBL learning

8.1 Supporting students in transitioning to active learning

One of the biggest challenges in IBL is that many students are used to passive instruction, where the instructor provides direct answers. Early on, I noticed that students were hesitant to explore concepts independently. To ease this transition, I start with structured inquiry activities before progressing to open-ended problem-solving, giving students time to adjust. I also explain from the beginning why IBL is beneficial and provide frequent feedback to keep them from feeling lost. Encouraging reflection through journals and discussions helps students track their growth and build confidence. The addition of a learning assistant has made a significant difference, offered peer support, and helped guide students through the inquiry process.

8.2 Preparing instructors for the shift to facilitation

For instructors accustomed to lecture-based teaching, shifting to IBL can feel uncertain. I initially struggled with letting go of direct instruction, fearing that students would not grasp key concepts. However, my IBL training helped me see that my role was not to provide answers but to guide discussions and prompt deeper thinking. Institutions should support instructors by offering professional development on managing active classrooms, facilitating discussions, and designing effective inquiry activities. Peer observations were especially helpful for me, as seeing IBL in action gave me practical strategies to implement. Having both an in-class tutor and a learning assistant in my classroom has also allowed me to focus more on facilitating inquiry while ensuring students receive extra support.

8.3 Fostering effective group collaboration

Collaboration is a cornerstone of IBL, but without structure, some students dominate discussions while others disengage. Early on, I realized that assigning roles such as discussion leader, recorder, or presenter helped balance participation and keep students accountable. Rotating these roles ensures that all students contribute meaningfully, and over time, they learn to manage discussions on their own. The learning assistant has played a key role in this process, stepping in to guide struggling groups, ask clarifying questions, and encourage quieter students to engage.

8.4 Balancing inquiry with course coverage

A common concern about IBL is that it takes more time, making it difficult to cover all required content. Initially, I faced this challenge myself, but I adapted by prioritizing the mathematical concepts that benefit most from inquiry while using direct instruction when necessary for procedural skills. In recent years, I have also posted instructional materials online and encouraged students to review them before class. While this is not a formal flipped classroom model, it has helped students come to class more prepared and ready to engage in deeper discussions. The addition of an in-class tutor and a learning assistant has further supported students by providing individualized assistance, ensuring they stay on track while maintaining an inquiry-driven approach.

8.5 Aligning assessments with IBL learning objectives

Traditional assessments often focus on procedural accuracy rather than conceptual understanding, making it difficult to measure the full impact of IBL. My training emphasized the importance of using a mix of formative and summative assessments, including problem-solving tasks, written reflections, and peer evaluations. Over time, I have refined my assessments to emphasize reasoning and conceptual understanding over rote memorization. Regular formative assessments, such as quizzes, discussions, and instructor feedback, allow students to refine their thinking, while summative assessments require them to apply concepts in meaningful ways. The Learning Assistant and in-class tutor have played an important role in this process by providing additional guidance and feedback, helping students articulate their reasoning more effectively.

9. Conclusion

This chapter explored the implementation of Inquiry-Based Learning (IBL) in mathematics education, highlighting its impact on student engagement, conceptual understanding, and problem-solving abilities. The findings from this study reinforce the idea that active learning approaches, particularly IBL, create dynamic and interactive learning environments where students take ownership of their education rather than passively receiving information. By shifting the instructional model from lecture-driven teaching to inquiry-driven learning, students develop stronger critical thinking skills, a deeper understanding of mathematical concepts, and greater confidence in their ability to tackle complex problems.

The results demonstrated that students in the IBL classroom exhibited higher engagement levels, greater retention rates, and improved problem-solving abilities compared to those in traditional lecture-based settings. This aligns with existing research emphasizing the effectiveness of inquiry-driven instruction in fostering meaningful learning experiences and long-term comprehension of mathematical concepts. Additionally, the integration of collaborative learning, formative assessment strategies, and real-world applications played a significant role in helping students bridge the gap between theoretical knowledge and practical application.

Despite its benefits, the implementation of IBL requires careful planning, structured support, and ongoing assessment to ensure student success. Challenges such as student resistance, instructor adaptation, and balancing inquiry with curriculum

coverage must be addressed through intentional instructional design, professional development, and strategic scaffolding. The recommendations outlined in this chapter provide a roadmap for educators seeking to adopt IBL effectively in their mathematics courses.

Ultimately, the shift toward inquiry-based instruction represents more than just a change in teaching methodology; it is a move toward a more student-centered, exploration-driven approach that equips learners with the skills necessary for success beyond the classroom. As mathematics education continues to evolve, embracing active learning frameworks like IBL will be crucial in preparing students to navigate an increasingly complex and analytical world. Future research should continue to explore best practices for scaling IBL across diverse learning environments, ensuring that all students, regardless of background, have access to engaging and effective mathematics instruction.

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Conflict of interest

The author declares no conflict of interest.

Declaration

This book chapter is based on the previously published article: Khasawneh et al. [60].

The article itself was derived from the dissertation: Khasawneh [61].

The content has been revised and adapted for inclusion in this book, with modifications to enhance readability, integrate additional insights, and align with the book's overarching theme. This chapter complies with copyright regulations and adheres to the ethical guidelines of academic publishing.

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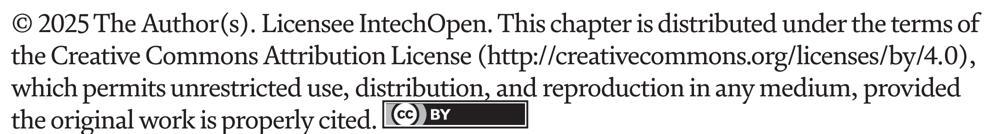
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Chapter 2

Perspective Chapter: Teacher and Classroom Influences on Mathematics Anxiety – A Systematic Literature Review of Instruction, Teacher Support, and Classroom Structure

Qian Li, Kexin Huang, Yipeng Hu, Shentian Ying, Ning Chen and Rosalie Palaroan

Abstract

The current systematic literature review synthesized findings from 56 empirical studies, which examined how different teacher- and classroom-level factors influence K-12 students' mathematics anxiety. Four major categories emerged from the analysis: (1) instructional practices and pedagogical strategies, (2) teacher support and classroom climate, (3) the structure of learning tasks and classroom routines, and (4) teachers' personal characteristics and beliefs. Research findings indicated that instructional strategies such as gamification and cognitive activation, as well as teachers' emotional and motivational support, are consistently associated with lower levels of student mathematics anxiety. In addition, some studies that compared multiple classroom-related factors found that instructional approaches focused on enhancing students' mathematics understanding were more effective than emotional regulation interventions in reducing math anxiety. Among the teacher-related factors, studies revealed that teachers' own mathematics anxiety and low self-efficacy in teaching are positively associated with students' mathematics anxiety. Guided by Control-Value Theory and Social Cognitive Theory, the review addresses how these teacher- and classroom-level factors shape students' mathematics learning experiences and provides evidence-based suggestions to inform relevant educational interventions.

Keywords: mathematics anxiety, teacher support, instruction, classroom climate, teacher-student relationship

1. Introduction

Mathematics anxiety, characterized by students' feelings of tension and apprehension in mathematics-related learning and testing situations, is prevalent among K-12 students and significantly hinders their mathematics achievement [1]. Research from diverse cultural backgrounds, such as the Program for International Student Assessment (PISA), has consistently shown that higher levels of mathematics anxiety are linked to lower performance [2, 3]. Besides the negative effect on academic achievement, mathematics anxiety also influences students' long-term academic and professional development and overall psychological well-being [4]. Accordingly, a growing body of studies has focused on developing strategies to mitigate its negative consequences. Intervention research has indicated that cognitive-behavioral therapy (CBT) can successfully help students regulate negative emotions connected to learning mathematics [5]. However, integrating CBT into classroom practice remains impractical due to limited resources and professional guidance. To provide a more practical and sustainable alternative, researchers have studied how teachers can change their teaching strategies and classroom routines to encourage students' positive experiences and attitudes toward mathematics.

Previous research has identified various teacher-level and classroom-level factors associated with students' mathematics anxiety, such as teacher support, teachers' own attitudes toward mathematics, and pedagogical strategies. For example, Li et al. [6] conducted a longitudinal study and revealed that higher levels of perceived teacher support predicted lower levels of mathematics anxiety. Furthermore, teachers' own emotional experiences with mathematics could be conveyed to students indirectly through their instructional practices and classroom management. McLean et al. [7] found similar results: that teachers' mathematics anxiety is positively correlated with their students' mathematics anxiety. The effect was also stronger among students from lower socioeconomic status (SES) backgrounds. Additionally, some researchers investigated the effectiveness of certain instructional interventions in reducing students' mathematics anxiety. For example, Alabdulaziz [8] integrated mathematics instruction into escape-room games and found a significant improvement in students' mathematics motivation and performance and a decline in mathematics anxiety.

The present study aims to systematically review empirical research published within the past 5 years that examined the association between teacher-level and classroom-level variables and K-12 students' mathematics anxiety. The objective of this systematic review is to identify and compare how different teacher- and classroom-related variables contribute to students' mathematics anxiety.

2. Literature review

2.1 Mathematics anxiety: A multidimensional construct

Mathematics anxiety refers to a negative psychophysiological reaction occurring in mathematics-related contexts [1]. Previous research has identified its multidimensional nature, including three main components: the emotional states (i.e., feelings of fear and nervousness), the cognitive appraisal (i.e., worries and concerns regarding the ability and performance), and the physiological arousal (e.g., increased heartbeat) [9]. These anxiety responses emerge in diverse mathematics-related contexts,

ranging from everyday numerical calculations to high-stakes testing situations. Additionally, Wigfield et al. [10] categorize mathematics anxiety into two distinct situational contexts: mathematics learning anxiety, experiences during instructional activities and problem-solving processes, and mathematics test anxiety, experienced specifically in evaluative or standardized testing environments.

Mathematics anxiety is highly prevalent among students across different educational stages and cultural contexts. Empirical evidence indicates that students' negative emotional responses toward mathematics can emerge as early as kindergarten, intensifying during the middle school years due to the increased pressures from high-stakes standardized assessments [11, 12]. Multiple meta-analyses have synthesized findings on the relationship between mathematics anxiety and achievement, consistently identifying a significant, moderate negative correlation [4, 9]. Students with greater mathematics anxiety typically exhibit reduced effort and engagement, heightened task avoidance, and diminished mathematics performance. Given these negative outcomes, it is critical to investigate factors contributing to mathematics anxiety and to develop targeted interventions or classroom strategies to enhance students' positive mathematics learning experiences.

2.2 Control-value theory and mathematics anxiety

Control-value theory, proposed by Pekrun et al. [13], provides a conceptual framework for understanding factors that influence students' experiences of academic emotions, including mathematics anxiety. According to this theory, two dimensions shape students' emotional responses in academic contexts: perceived control, which refers to students' competence beliefs in successfully performing mathematics-related tasks or tests (e.g., "Can I solve this math problem?"); and perceived value, which involves students' perception of the intrinsic interest or practical usefulness of learning mathematics. Specifically, when students perceive higher levels of control over mathematical tasks, they are less likely to experience anxiety or stress. Similarly, perceiving mathematics as inherently valuable or practically useful enhances students' enjoyment and reduces anxiety associated with mathematics learning.

Several empirical studies have examined how different instructional practices influence students' mathematics anxiety by affecting their perceptions of control and value. For instance, Fergus and Smith [14] demonstrated that timely and formative feedback from teachers could help students adjust their understanding promptly, which in turn could enhance their perceived control and reduce their mathematics anxiety. Additionally, research indicates that emotional support and encouragement from teachers significantly contribute to a supportive classroom climate, promoting students' positive emotional experiences and attitudes toward mathematics and ultimately lowering their mathematics anxiety [15]. Overall, these findings revealed the critical role of teacher support and classroom practices in shaping students' control-value appraisal of mathematics learning, which further influences students' mathematics anxiety.

2.3 Social cognitive theory and mathematics anxiety

Social cognitive theory, originally proposed by Bandura [16], emphasizes the reciprocal interactions among individual cognition, behavior, and environmental factors. Within the educational context, the theory highlights the significant role

of teachers as models whose own attitudes, beliefs, and behaviors can influence students' learning process, such as their self-efficacy and mathematics anxiety. When teachers exhibit positive attitudes toward mathematics, show resilience when encountering challenges, and openly model effective problem-solving strategies, students can imitate the coping behaviors, develop a stronger self-efficacy in mathematics, and experience lower levels of anxiety in mathematics learning [17]. Additionally, teachers with high teaching self-efficacy tend to provide more autonomy support, less controlling language, and integrate diverse instructional methods in their classrooms. These instructional practices collectively contribute to students' positive emotional experiences in mathematics learning. Conversely, when teachers themselves hold a negative attitude toward mathematics, such as experiencing mathematics anxiety, they are more likely to show avoidance behaviors, nervousness, or fear toward mathematics [18]. In this case, students are likely to observe these verbal and nonverbal cues from teachers and reinforce students' own negative perceptions of mathematics.

Empirical evidence has consistently supported the influence of teachers' own attitudes and emotional reactions on students' mathematics anxiety and achievement. McLean et al. [7], for example, investigated mathematics anxiety among fourth-grade teachers and their students, identifying a significant positive association between teachers' and students' anxiety levels. Similarly, Richland et al. [19] and Kaskens et al. [20] provided evidence that mathematics anxiety experienced by teachers not only predicts students' negative attitudes toward mathematics but also directly influences their mathematics performance outcomes. Taken together, these studies highlighted the critical role of teachers' emotional and motivational characteristics in influencing students' mathematics anxiety.

2.4 Present study

Though a considerable number of studies have focused on the relationship between students' mathematics anxiety and various teacher-level and classroom-level factors, findings have been inconsistent. For instance, while Alabdulaziz [8] reported a significant decrease in students' mathematics anxiety after integrating gamified elements into mathematics learning, Yıldız and Yaman [21] found no significant effects among the gifted students. These inconsistencies highlight the need for a comprehensive and systematic literature review to clarify the influence of different teacher-related and classroom-related variables. Accordingly, this review aims to synthesize existing evidence on the associations between teacher- and classroom-level factors and students' mathematics anxiety to inform evidence-based educational interventions.

3. Methods

The literature search was conducted within two primary databases: PsycInfo and Education Resources Information Center (ERIC). The keywords for the literature search included "math anxiety, OR mathematics anxiety" AND "classroom OR teaching OR instruction." The search aimed to identify relevant peer-reviewed journal articles that were published within the past 5 years (from January 2020 to April 2025). The literature search initially retrieved 403 articles from the two databases (182 from PsycInfo and 221 from ERIC). After removing 50 duplicate articles, 353 studies were

reviewed for eligibility to be included in the current literature review. **Figure 1** shows the process of literature search and screening.

The literature screening process involved two stages, guided by the predefined inclusive and exclusive criteria shown in **Table 1**. During the initial screening round, the authors reviewed the titles and abstracts of 353 articles and excluded nine studies lacking empirical evidence (e.g., theoretical papers, editorial opinions) and 85 studies that did not focus on K-12 student populations (e.g., studies measured college students, pre-service teachers, or teachers' mathematics anxiety). Following the preliminary screening, the authors reviewed the full text of 259 articles and excluded the following articles: (1) 22 studies lacking empirical evidence, (2) 36 studies that did not primarily examine the K-12 samples, (3) 35 studies without a focus on mathematics anxiety, (4) 109 studies that did not include any teacher-related or classroom-related measures, and (5) one study lacking full text access after contacting the corresponding author.

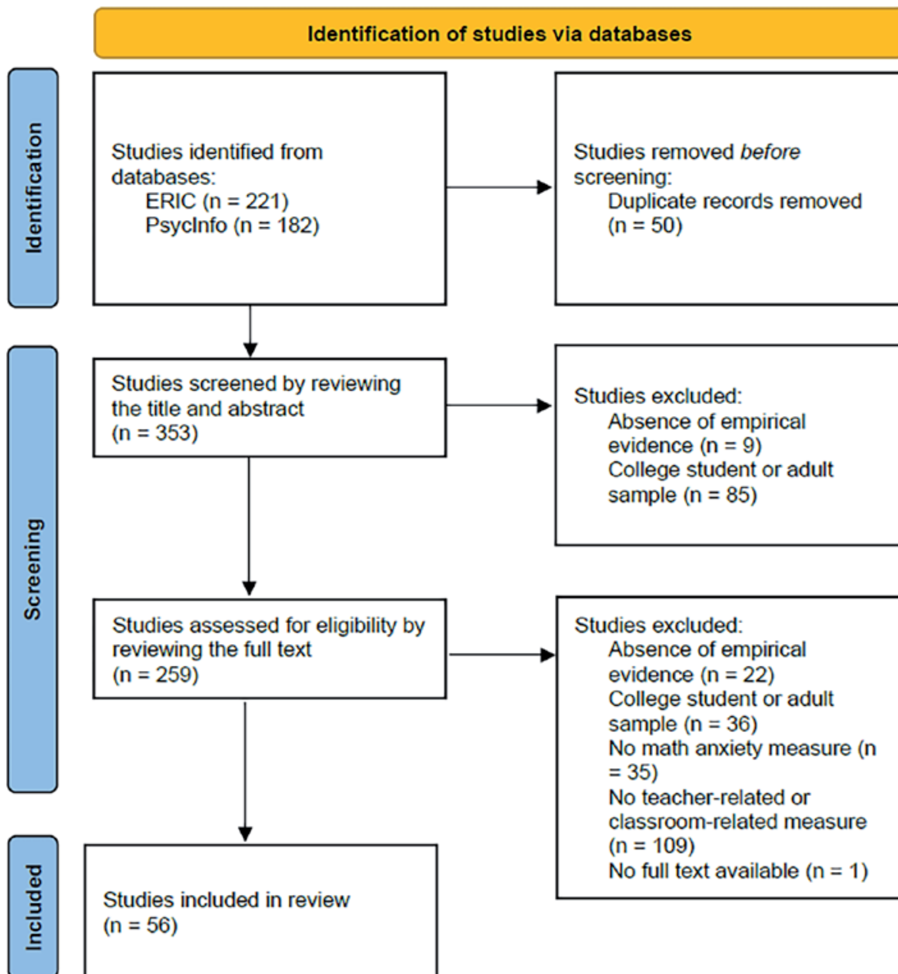

Figure 1.
Process of literature search and selection.

Criteria	Inclusion criteria	Exclusion criteria
Publication date	Between January 2020 and April 2025	Before January 2020
Access	Full text is available.	Full text is not available after contacting the corresponding author.
Research method	Included empirical evidence.	Lack of empirical evidence (e.g., theoretical article, editorial opinions, literature review, meta-analysis).
Sample	K-12 students	College students or adult sample (e.g., college students, pre-service teachers, teachers, parents)
Key construct	Included measures of mathematics anxiety	Lack of measures of mathematics anxiety
Key construct	Included measures of teacher-related or classroom-related variables	Lack of measures of teacher-related or classroom-related variables

Table 1.
Inclusion and exclusion criteria for literature screening.

4. Results

4.1 Description of included studies

A total of 56 studies met the inclusion criteria and were included in the current systematic literature review. After carefully reviewing the teacher-related and classroom-related constructs addressed within these studies, four major thematic categories emerged, as shown in **Table 2**. The first category is instructional practices and pedagogical strategies, examined in 24 studies, which refer to teachers' diverse approaches to delivering the course materials and organizing classroom activities to engage students with mathematics learning. Examples of instructional practices included incorporating gamified elements, cognitive activation methods, dialog-based learning activities, and the use of formative assessments. The second category is teacher support and classroom climate (from 22 studies). This category involves

Category	Label	Description	Examples
Category-1 (Identified from 24 studies)	Instructional practices and pedagogical strategies	Approaches to delivering content and organizing activities to support math learning.	Gamification, cognitive activation, dialog-based learning activities, and formative assessments.
Category-2 (Identified from 22 studies)	Teacher support and classroom climate	Teachers' support that fosters students' motivation and emotional well-being.	Encouragement, autonomy support, and positive teacher-student relationships.
Category-3 (Identified from six studies)	Structure of learning tasks and classroom routines	Strategies for managing classrooms and structuring learning tasks.	Rules and routines, task clarity, appropriate challenge, and workload management.
Category-4 (Identified from nine studies)	Teacher characteristics	Teachers' personal and professional traits influence student outcomes.	Teachers' mathematics anxiety, self-efficacy, and professional knowledge.

Table 2.
Major categories of teacher- and classroom-level variables.

how teachers support students' motivation and emotional experiences and provide a safe and positive learning environment. To be more specific, the relevant constructs included teacher-student relationships, perceived teacher support, autonomy support, teachers' motivational style, and emotional regulation interventions. The third category is the structure of learning tasks and classroom routines, which emerges from six studies and refers to teachers' strategies for classroom management (e.g., establishing rules and routines) and organizing instructional tasks (e.g., setting task demands, requirements, and levels of difficulty). The last category contains teachers' own characteristics, including teachers' own mathematics anxiety, teaching self-efficacy, and professional knowledge. Nine studies focused on how these teachers' personal traits or beliefs were associated with students' mathematics anxiety.

Among these included studies, five examined and compared multiple aspects across the thematic categories. For example, Passolunghi et al. [22] implemented two different interventions among fourth graders: one of the interventions focused on instructional training to improve students' use of calculation strategies (Category-1), while another intervention focused on emotional regulation training to help students reduce their negative feelings related to mathematics (Category-2). Results indicated that the instructional intervention was more effective in reducing students' mathematics anxiety and improving their mathematics performance. Similarly, Lu et al. [23] found that both the cognitive activation approach (Category-1) and perceived teacher support (Category-2) were negatively associated with eighth-grade students' mathematics anxiety, while the instructional support, in this case, the cognitive activation approach, demonstrated a stronger effect size.

4.2 Instructional practices and pedagogical strategies

Among the studies focused on instructional practices and pedagogical strategies, six studies have examined the effectiveness of incorporating gamified elements into instruction. Most studies (n = 5) reported that gamified instruction can significantly improve mathematics learning outcomes. For example, teaching methods incorporating escape rooms and digital games effectively stimulated students' interest and participation through random questions, timed challenges, and point rewards, and reduced students' mathematics anxiety [8, 24, 25]. Students reported that the gamified elements of instruction make mathematics more enjoyable and accessible. Besides the positive effect on students' interest in mathematics, researchers argued that gamification can integrate abstract knowledge into real situations and use adaptive technology to meet personalized needs in order to help students master knowledge more intuitively and support students' cognitive engagement [26]. Consequently, previous studies have consistently found that gamification can also support students' mathematics performance [8, 24, 26]. It is worth noting that the competitive and cooperative mechanisms in the game can promote peer interaction and form a benign learning atmosphere [24]. According to Karademir and Saatcioglu [27], it is important to use gamification strategies to cultivate a desire for exploration, especially in early education, rather than emphasizing academic training in early grades. However, the effects of gamification on mathematics anxiety may vary across the student population. For instance, Yıldız and Yaman [21] found that though gamification enhanced learning engagement, it did not significantly reduce mathematics anxiety among gifted students.

Second, 10 out of 24 studies addressed different elements of cognitive activation methods. According to Liu et al. [28], the cognitive activation teaching method

includes three major aspects: promotion of conceptual understanding, cognitive demand of students' activities, and quality of interaction and participation. Multiple studies directly examined students' perceptions of their teachers' use of cognitive activation methods and found its significant association with increased self-efficacy and decreased mathematics anxiety [23, 25, 28]. Using an intervention design, Morsanyi et al. [29] examined the effectiveness of cognitive activating instruction, showing that process-based sorting training improved early mathematical skills and reduced anxiety. The approach integrated teacher scaffolding, developmentally appropriate manipulatives, and adaptive task parameters to accommodate individual differences. Another line of studies focused on the conceptual understanding aspects and revealed the effect of cognitive activation on mitigating students' mathematics anxiety. Six of the studies focused on different approaches to support students' conceptual understanding in mathematics. For example, several studies focused on how to connect mathematical concepts and knowledge with daily life applications through different approaches, including incorporating mathematical concepts into historical stories [30], cultural scenarios [31], and real-life applications [27]. These studies suggested that these real-life connections can support students' interest in mathematics and change the students' common bias that "math is boring and not useful." Other instructional strategies included dialog-based learning activities, conceptual teaching, and elaborated instruction using sample problems, all designed to facilitate extended discussions and clarify abstract mathematical concepts and formulas [32–34]. Research findings consistently indicated that when students developed a clearer understanding and structure of the mathematics knowledge, they reported high engagement in mathematics learning and lower negative feelings related to mathematics.

Furthermore, the roles of formative assessment practices and formative feedback approaches have been addressed in three articles. Formative assessment practices effectively reduce mathematics anxiety by enabling students to promptly adjust their understanding and enhance their perceived control, including performance evaluation and feedback in online math games [35, 36]. For instance, Yang and Lu [36] demonstrated that integrating two-tier testing mechanics within educational games, which assess answer accuracy and diagnose underlying misconception types, significantly enhances learning outcomes, particularly for students who actively engage with the feedback provided. Simultaneously, formative feedback approaches such as clear learning objectives and assessment for learning [14] have also been shown to effectively reduce mathematics anxiety while enhancing learning quality.

Moreover, one study applied the Self-Regulated Strategy Integration model and demonstrated dual cognitive-affective benefits by systematically teaching self-regulatory learning strategies [37, 38]. The research indicated that the Self-Regulated Strategy Integration model enhances academic performance while reducing anxiety, a pattern empirically validated by Pizzie and Kraemer [37]. Their work showed that structured habit formation (e.g., scattered review and retrieval practice) not only mitigates mathematics avoidance behaviors but also buffers against anxiety's detrimental effects on achievement.

4.3 Teacher support and classroom climate

Teacher support and classroom climate have emerged as crucial factors influencing mathematics anxiety. Among the 56 articles, 22 explored the association between teacher support and mathematics anxiety. These studies consistently reported that

high levels of perceived teacher support, encouragement, effective motivational styles and strategies, appropriate teacher expectations, and high-quality teacher-student relationships can significantly mitigate students' mathematics anxiety. Teacher support typically includes multiple components, including autonomy support, cognitive support, and emotional support [39]. It can not only support students' psychological needs for autonomy, relatedness, and competence [40] but also act as a stress buffer when they face difficulties [23]. For example, encouraging feedback in mathematics learning could inform students about their learning goals and progress, as well as support students' positive emotional experiences, thus mitigating their mathematics anxiety [14]. According to Lee and Ward-Penny [41], with these three types of support, teachers can help students develop higher levels of resilience in mathematics learning, feel more in control of the mathematics learning process, and affirm the value of mathematics, thereby enhancing their self-efficacy and reducing their mathematics anxiety.

More specifically, Yang et al. [42] focused on teachers' motivational style: the controlling style and the autonomy-supportive style. The former is teacher-centered and adheres to highly structured instruction in the context of Communicative Language Teaching (CLT), while the latter is student-centered with less structured instruction. Among students with a high conceptual level, there are no obvious differences between the two motivational styles because these students have lower mathematics anxiety, clearer goals, and stronger motivation for mathematics learning. For students with a low conceptual level, the controlling style is more effective in moderating their mathematics anxiety. Thus, an appropriate match can positively moderate the relationship between conceptual level and mathematics anxiety. According to Yang et al. [42], although generally considered counterproductive, a controlling motivational style can reduce mathematics anxiety when balanced with autonomy support. For example, when teachers guide students' critical reasoning to help them understand the connections between mathematical concepts while allowing students the freedom to choose problem-solving methods, this provides students with direction and clear goals and enhances their belief in controlling their learning. In this way, when students successfully reach the goals set by teachers, their self-efficacy is boosted and mathematics anxiety is reduced.

Moreover, the effect of the teacher-student relationship is examined in multiple studies. The teacher-student relationship refers to the connection and interaction between teachers and students [43]. The teacher-student relationship is related to a series of academic behaviors, such as academic engagement, performance, and motivation, and it further influences emotional regulation, self-concept, and academic achievement [41]. Generally, students in a high-quality teacher-student relationship tended to experience lower levels of mathematics anxiety, as they perceived trust and safety, received emotional support and personalized guidance, engaged in a positive learning climate, and developed self-efficacy. Especially in the context of mathematics anxiety, a positive teacher-student relationship helps alleviate mathematics anxiety, enhances students' confidence in solving math challenges, and can provide personalized guidance. Even for students at risk of school failure, such a relationship acts as a protective factor and positively guides them [15]. However, students with poor teacher-student relationships tend to exhibit higher levels of aggressive behavior and have a negative self-concept. Undoubtedly, they find it more difficult to make progress in mathematics learning, which is directly correlated with high mathematics anxiety [44]. Moreover, rural students are more profoundly affected [45].

A positive instructional climate also contributes significantly to the quality of teacher-student interaction and can act as an emotional buffer for students facing mathematical challenges. Emotionally engaging teaching practices, such as interactive activities and student-led task selection, not only foster stronger interpersonal connections but also increase students' sense of autonomy and psychological safety in the classroom [24, 46]. These practices reflect a motivational teaching style that values students' emotional needs and supports intrinsic motivation. Especially in early education settings, teachers who create playful, supportive learning environments can reduce math-related stress while promoting curiosity and emotional engagement. According to Naseem [47], a nationwide climate of educational pressure often amplifies student anxiety, but warm and confident instructional approaches help buffer against this effect. By providing a secure, low-pressure atmosphere, teachers encourage positive emotional regulation and help students develop a stronger sense of self-efficacy. Thus, the instructional climate not only reinforces the emotional dimension of teacher-student relationships but also shapes students' attitudes toward mathematics through daily classroom experiences.

4.4 Task and classroom structure

Among the 56 studies included in the current review, six studies specifically highlighted the impact of tasks and classroom structure. To be more specific, classroom-level factors such as classroom management, task design and clarity, appropriate challenge, and workload management collectively contribute to a well-structured learning environment that supports students' psychological needs like autonomy, competence, and relatedness, which are key elements that enhance students' intrinsic motivation and reduce anxiety.

First, clear and effective classroom organization, including well-defined learning objectives, structured routines, and consistent teacher support, creates a predictable environment that enhances students' emotional security and academic engagement. For instance, Chan and Liem [25] found that structured guidance significantly increased motivation among high-achieving students, and another study [48] demonstrated that effective classroom management skills predicted student performance in math, contributing to the reduced level of mathematics anxiety. A well-managed classroom could minimize distractors, help students concentrate, and boost their sense of control over learning tasks. This, in turn, fosters emotional confidence, which has been linked to higher achievement and lower anxiety in mathematics.

Second, task difficulty also plays a crucial role in shaping students' math experiences. Research indicates that the association between task difficulty and students' positive learning experiences is not a simple linear relationship: overly easy tasks may reduce engagement, while overly hard ones can induce frustration and anxiety. Instead, well-designed tasks should be challenging yet achievable, offering educational relevance and appropriate scaffolding. Bilgin [24] showed that such tasks could increase interactivity and reduce mathematics anxiety. In addition, Samuelsson [46] emphasized that meaningful, real-life-related challenges could support students' intrinsic motivation and learning experience.

Third, previous studies have emphasized the important role of workload management. Qin and Wang [49] reveal a U-shaped relationship between homework time and mathematics performance: while around 45 minutes of daily math homework optimizes achievement, reducing it to 35 minutes can foster greater interest. Excessive time demands, especially under high pressure, can intensify mathematics anxiety,

particularly among low-motivated students. Time constraints during tests are another significant source of stress [50]. Therefore, the structure of both the learning task and classroom routines could contribute to students' emotional experiences related to mathematics.

4.5 Teacher characteristics and beliefs

Previous studies have consistently revealed a significant and positive association between teachers' and students' mathematics anxiety, typically with a small to moderate effect size. For example, Szczygieł [51] found a direct relationship between teachers' and students' mathematics anxiety among elementary students across grades one to three. Similarly, Richland et al. [19] reported that teachers' mathematics anxiety significantly, though weakly, predicted elementary students' mathematics anxiety. Schaeffer et al. [18] further demonstrated that higher levels of teacher mathematics anxiety not only predicted increased mathematics anxiety in students but also led to lower academic performance. Interestingly, research has shown that while teachers' math anxiety may not be related to students' achievement at the beginning of the school year, a negative association often emerges by the end of the academic year [18, 19, 52]. Researchers argued that teachers' mathematics anxiety can be both directly transmitted to students and indirectly influence students through the changes in their teaching behaviors [51]. Teachers' own confidence, knowledge, and satisfaction with teaching can also influence students' mathematics learning experiences and, in turn, affect their levels of mathematics anxiety. Research indicated that teachers' self-efficacy and professional knowledge in teaching are significantly associated with students' mathematics anxiety [20, 53].

5. Discussion

The current systematic review synthesized evidence from 56 empirical studies investigating the association between teacher-level and classroom-level variables and students' mathematics anxiety. Through a structured analysis, four distinct categories emerged: (1) instructional practices and pedagogical strategies, (2) teacher support and classroom climate, (3) classroom and task structure, and (4) teachers' personal characteristics. The mechanisms underlying the relationships between these teacher- and classroom-related factors and mathematics anxiety can be effectively explained through the frameworks provided by Control-Value Theory and Social Cognitive Theory.

First, from the perspective of Control-Value Theory [13], students' academic emotions, including mathematics anxiety, are primarily shaped by their perceived sense of control over learning outcomes (e.g., self-efficacy) and their subjective value attached to the academic tasks (e.g., intrinsic interest, perceived relevance). Previous studies have consistently demonstrated that various instructional practices and pedagogical strategies directly influence these two critical cognitive appraisal dimensions and affect students' anxiety levels. For instance, several reviewed studies highlighted the effectiveness of gamified instructional methods [8, 24, 26]. Incorporating game-based elements, such as competitive activities, point systems, and immediate feedback, enhanced students' intrinsic enjoyment and curiosity in mathematics. This heightened intrinsic value, in turn, lowered mathematics anxiety by fostering positive emotional experiences. Similarly, instructional strategies that explicitly connected

mathematics content with real-world applications successfully elevated students' perceived relevance and practical value of mathematical knowledge, thus reducing anxiety through increased task meaningfulness [32–34].

Besides the influence on students' perceived value of mathematical knowledge, instructional strategies examined in this review focused on supporting students' sense of control and competence in mathematics. For example, multiple studies have applied cognitive activation methods to improve students' conceptual understanding in mathematics learning [23, 25, 28]. By promoting students' active engagement and conceptual understanding of mathematics concepts, these methods increased their mathematics self-efficacy, subsequently mitigating anxiety. Through such instructional approaches, mathematics instruction becomes less abstract and more contextualized, allowing students to feel greater control and competence in their learning process.

Similarly, the review revealed that teacher socio-emotional support and structured classroom environments significantly mitigate mathematics anxiety by influencing students' control and value appraisals. Research that addressed teacher support and classroom climate emphasized the critical role of emotional support, encouragement, autonomy-supportive practices, and positive teacher-student rapport [44–46]. Such supportive practices fulfill students' psychological needs for safety, belonging, and emotional reassurance, resulting in enhanced positive attitudes toward mathematics and reduced anxiety levels. Moreover, clearly structured classroom routines, predictable instructional practices, and appropriately challenging academic tasks create environments conducive to student confidence and control perceptions. Clear task structures reduce cognitive overload and uncertainty, thereby decreasing anxiety. Although research in this particular category remains somewhat limited, existing findings align consistently with predictions from Control-Value Theory.

The impact of teachers' personal characteristics in shaping students' mathematics anxiety can be explained through the perspective of Social Cognitive Theory [16]. According to this theory, significant role models in students' social settings, including their teachers, have a strong influence over their cognitive, emotional, and behavioral processes. Studies included in the current review consistently pointed out that teachers' personal characteristics, such as their own mathematics anxiety, teaching self-efficacy, and professional knowledge, can be both explicitly and implicitly communicated to students. For instance, highly mathematics-anxious teachers can unconsciously convey their negative attitude and avoidance behaviors in teaching, which can lead students to internalize related anxious attitudes toward mathematics. Therefore, anxious or less confident students may learn enhanced anxiety and mathematics-negative attitudes from teachers through observation, learning, and modeling processes.

6. Conclusions

In summary, this systematic literature review provided a comprehensive synthesis of empirical evidence regarding teacher-level and classroom-level predictors of mathematics anxiety. The findings are consistent with the theoretical claims from Control-Value Theory and Social Cognitive Theory. By understanding these theoretical mechanisms, teachers can develop targeted instructional strategies and interventions to reduce students' mathematics anxiety. Future research should further examine mediational mechanisms and interactive relationships among the

identified teacher- and classroom-related variables, particularly emphasizing effective approaches to professional development and instructional design. Such research could provide actionable insights to better support students' academic achievement and psychosocial well-being in mathematics education contexts.

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Conflict of interest

The authors declare no conflict of interest.

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Section 2

Conceptual Clarity and Theoretical Depth in Mathematics Education

Chapter 3

Perspective Chapter: Mathematical Definitions – A Key to Conceptual Clarity and Theoretical Rigor

Maryam Alkandari

Abstract

This chapter presents the Conceptual Definition Model (CDM) as a comprehensive and structured framework for enhancing the teaching and learning of mathematics through a focus on definitional clarity. The model underscores the pivotal role of precise, logically grounded definitions in cultivating conceptual understanding and theoretical coherence. CDM integrates elements such as logical structure, illustrative examples, counterexamples, and the articulation of necessary and sufficient conditions to help students navigate mathematical meanings more deeply. Rather than treating definitions as static or memorized statements, this model promotes their use as dynamic cognitive instruments that guide reasoning and inquiry. By engaging students in activities such as dissecting definitions, testing edge cases, and constructing definitions in reverse, CDM fosters analytical thinking and ownership of knowledge. The chapter argues for a paradigm shift in mathematical pedagogy—from passive assimilation to active construction of meaning—thereby empowering learners to participate meaningfully in the logical architecture of mathematics. The educational value of this model is further emphasized through its potential to stimulate reflective questioning and resolve cognitive conflict, ultimately improving proof comprehension and logical reasoning.

Keywords: mathematics education, concept image, cognitive conflict, proof comprehension, logical reasoning, necessary and sufficient conditions, active learning

1. Introduction

Mathematical definitions represent both the bedrock and the bottleneck of advanced mathematics education. While they are indispensable for establishing formal rigor and logical coherence, they frequently act as cognitive obstacles rather than intellectual entry points. Empirical studies reveal a persistent disjunction: although approximately 80% of university-level mathematics students can recite formal definitions, fewer than 30% demonstrate the ability to apply them effectively in proofs or problem-solving contexts [1, 2]. This stark contrast underscores a pedagogical fault line between memorization and conceptual mastery—an issue that continues to challenge mathematics instruction despite extensive curricular innovations.

Traditional instructional practices often treat definitions as fixed declarations to be absorbed and reproduced rather than as dynamic instruments of reasoning and exploration. For instance, the definition of a prime number typically omits any discussion of why the number 1 is excluded—an omission with profound implications for the structure of number theory, particularly the Fundamental Theorem of Arithmetic [3]. This passive transmission aligns with what Dubinsky [4] describes as the ritualization of knowledge, whereby students emulate formalism without engaging its conceptual foundations. The consequences are far-reaching: in upper-division courses such as real analysis, nearly two-thirds of student errors stem not from procedural missteps but from conceptual misunderstandings rooted in definitional ambiguity [5].

1.1 CDM introduction paragraph (excerpt)

In response to persistent challenges, this chapter introduces the Conceptual Definition Model (CDM)—a framework that definitions as dynamic cognitive tools. Neurocognitive evidence [6] confirms that active definition construction engages deeper learning pathways than rote memorization. The CDM's five-phase process, including reverse engineering [7] and edge case analysis [8], addresses the 62.3% of errors stemming from definitional ambiguity [5].

Drawing on foundational theoretical perspectives - Vinner's (1991) concept image theory highlighting the divergence between formal definitions and students' mental representations; Tall's [9] three worlds of mathematics, particularly the formal-axiomatic world emphasizing logical structure; and Watson and Mason's [10] work on example spaces stressing counterexamples - the CDM integrates these frameworks into a cohesive five-phase process for deeper analytical understanding.

1. Definitional deconstruction: Students dissect definitions into their necessary and sufficient conditions, promoting clarity in logical structure and enhancing their ability to distinguish essential properties.

2. Contrastive example analysis: Grounded in Watson and Mason's notion of example spaces, this phase engages students in comparing well-chosen examples and non-examples to clarify conceptual boundaries and reveal misconceptions (e.g., [8]).

3. Edge case interrogation: Students explore ambiguous or marginal cases—such as zero, negative integers, or atypical geometric shapes—to evaluate the robustness and inclusiveness of definitions [10].

4. Reverse engineering: Drawing on constructivist principles, students generate or revise definitions by identifying structural patterns in valid instances, cultivating inductive reasoning and ownership of knowledge [7].

5. Biconditional validation: Reinforcing logical accuracy, students assess whether the definition satisfies both necessity and sufficiency, aligning their reasoning with formal mathematical standards [9].

Through these five interdependent phases, the CDM transforms traditional instruction into an active process of conceptual inquiry, promoting critical thinking, metacognitive reflection, and flexible understanding [11].

While grounded in established theories, the CDM constitutes an original instructional design that reorganizes these insights into a coherent, process-based framework, specifically tailored to address enduring gaps in students' conceptual understanding and to empower learners as co-constructors of mathematical meaning.

In abstract algebra settings, classrooms implementing CDM observed a 40% improvement in proof accuracy, particularly when applying definitions such as subgroup and homomorphism [12]. These findings support Bruner's [13] assertion that "knowing is a process, not a possession" (p. 72), challenging the assumption that mathematical rigor is synonymous with rote learning [14].

This chapter contributes to mathematics education through three key dimensions:

- Theoretical contribution: By integrating Vygotsky's [15] sociocultural theory with formal logic, the chapter repositions definitions as tools of mathematical culture and cognition.

- Pedagogical contribution: It introduces instructional techniques such as *Definitional Stress Tests* and *Conceptual Boundary Mapping* to scaffold student understanding.

- Empirical contribution: It presents cross-disciplinary data supporting the effectiveness of CDM across mathematical domains, including geometry, algebra, and analysis.

Rather than simplifying mathematical ideas, the CDM framework renders their logical architecture more visible and accessible, transforming definitions from inert content into active sites of mathematical thinking.

1.2 The core argument of the chapter

This chapter advances the central claim that learning mathematics is fundamentally a process of learning through definitions. Far from being peripheral components of mathematical discourse, definitions constitute its cognitive infrastructure—they are the frameworks through which mathematical thought is structured, communicated, and expanded.

Traditionally, definitions are presented as static statements to be memorized, rendering them passive and inert in students' minds. However, when learners engage with definitions as active conceptual tools, they cultivate the essential dispositions of the discipline: precision, abstraction, logical rigor, and critical inquiry. This epistemological shift—from rote memorization to active meaning-making—marks a movement from mathematical literacy (the ability to follow procedures) to mathematical fluency (the capacity to think, reason, and construct within the discipline).

Such a transformation enables students to transcend mechanical rule-following and participate in the generative logic of mathematics. They begin to treat definitions not merely as inputs to be recalled, but as objects of analysis and construction—interrogating their boundaries, testing their scope, and reconstructing them in response to conceptual challenges. In doing so, they enter the epistemological core of mathematics, where definitions are both scaffolding and engine—the foundation upon which proofs are built, and the mechanism by which abstraction evolves.

By reimagining definitions as dynamic and participatory, this chapter argues for a pedagogical reorientation—one that positions students as co-constructors of

mathematical meaning and empowers them to engage with mathematics as a creative, structured, and intellectually demanding enterprise.

2. Development

> *"Definitions are the DNA of mathematics – they carry within them not just what we know, but how we come to know it."*

In the architecture of mathematical thought, definitions are more than mere linguistic labels; they are the foundational blueprints that construct mathematical meaning. Just as DNA encodes the instructions for life, mathematical definitions encode the intellectual pathways by which concepts are formed, relationships explored, and truths verified. They are not static facts to be memorized but living structures that shape the very act of knowing in mathematics [16].

2.1 The historical and epistemological role of definitions in mathematics

2.1.1 From description to construction

Historically, mathematical definitions evolved from passive descriptors to active constructors of mathematical reality. In *Euclid's Elements*, the definition of a point as "that which has no part" did not merely describe it—it created a foundational entity upon which an entire deductive system was built [11]. This marked mathematics' first definitional revolution: recognizing that definitions do not follow mathematics—they generate it

2.1.2 The nineteenth century: Definitions as engines of abstraction

A second revolution emerged in the nineteenth century, driven by the rise of abstract algebra. Mathematicians such as Cayley and Noether no longer aimed to define concepts by appealing to intuition. Instead, they used definitions to liberate mathematics from its empirical roots:

- Groups were redefined beyond permutations into abstract algebraic systems.
- Fields generalized number systems.
- Rings unified arithmetic and algebraic ideas [17].
- These were not semantic adjustments but ontological shifts—definitions reimagined what mathematics could be.

2.1.3 Set theory and the paradox of precision

The development of set theory illustrates the paradoxical power of definitions. Cantor's intuitive definition of a set as a "collection of distinct objects" led to Russell's famous paradox, exposing a critical insight: definitions must balance rigor with flexibility. The resulting ZFC axiomatic system reflects the height of definitional maturity—simultaneously constraining inconsistency and enabling discovery [14].

2.1.4 The evolution of "function"

The concept of function provides a clear case study of definitional evolution:

- Eulerian Era: Functions as algebraic expressions.
- Dirichlet's Leap [18]: Functions as arbitrary input-output relations [19].
- Modern Formalism: Functions as set-theoretic mappings (Bourbaki).

Each step resolved previous limitations while broadening the concept's applicability. Definitions here are revealed not just as constraints, but as launchpads for new mathematical realities [20].

2.1.5 The creative paradox

Modern mathematics embodies a compelling irony: the more abstract and precise definitions become, the greater their generative power. While Euclid's definitions described one geometry, today's axiomatic systems define countless mathematical "universes." Definitions have thus transitioned from boundaries to building blocks—not restricting thought, but constructing it.

2.2 Theoretical background

Our pedagogical framework draws from logic theory, cognitive psychology, and sociocultural learning models. Influenced by Bruner's [13] theory of concept attainment and Vygotsky's [15] ideas on mediated learning, we view definitions as structured cognitive tools. A mathematical definition is not a label but a logical container—a precise statement of necessary and sufficient conditions, often expressed using the "if and only if" structure [14].

However, there remains a troubling disjunction between formal definitions and students' understanding. Vinner [1] distinguished between "concept definition" and "concept image"—showing that many students rely on informal visual or experiential cues rather than the formal logic of definitions. This gap has significant consequences. In one study, 96% of university students could not properly define the concept of a function, impairing their ability to move between algebraic, graphical, and verbal representations [21]. Similar findings by Zaslavsky and Shir [2] show that students often treat definitions as vague descriptions instead of rigorous tools.

2.2.1 Research-based pedagogical strategies for teaching mathematical definitions

Recent research in mathematics education has emphasized the importance of instructional strategies that promote students' deep understanding of definitions rather than mere memorization. The following pedagogical approaches, supported by empirical and theoretical studies, have been shown to enhance students' conceptual engagement with mathematical definitions:

1. Reverse engineering definitions

   As shown by *Arthur and Dagan [7]*, definition reconstruction enhances students' inductive reasoning in discrete mathematics. Also *Larsen and Lockwood [12]*

highlight the effectiveness of having students reconstruct mathematical definitions based on examining multiple examples and identifying common properties. This method encourages learners to reason inductively and develop a deeper understanding of what makes a definition both necessary and sufficient.

- Example: Learners may explore group elements and cosets before formally constructing the definition of a normal subgroup, fostering active discovery rather than passive reception.

2. Definitional reasoning through counterexamples

   Vroom et al. [22] argue that engaging students in testing and refining definitions by generating counterexamples and edge cases helps them sharpen the logical boundaries of mathematical concepts.

   - Illustration: Questions like "Is a point a circle with radius zero?" are used to challenge students' initial assumptions and refine conceptual clarity.

3. Visual scaffolding

   Fernández-Plaza et al. [8] emphasize the use of visual tools such as concept maps, diagrams, and dynamic software environments to make abstract definitions more accessible. These tools help students see the structure and interrelations among concepts, aiding their comprehension and retention.

   - Example: Using dynamic geometry software to illustrate the definition of continuity via ε-δ conditions provides students with both formal rigor and intuitive insight.

4. Recursive conceptual construction

   Arthur and Dagan [7] discuss strategies where students build complex definitions from simpler components, often using recursive reasoning or inductive patterns. This promotes a layered understanding of advanced topics and aligns with how mathematical knowledge is constructed in higher education.

2.2.2 Synthesis and theoretical implications

The convergence of findings from these studies reveals a powerful pedagogical insight: students master definitions more effectively when they are actively involved in their construction, critique, and application. This pedagogical shift parallels the historical evolution of core mathematical ideas—for example, the conceptual refinement of "function" from intuitive usage to formal definition.

Definitions should not be treated as final products to memorize but rather as cognitive tools that structure thought, guide inquiry, and support creativity. They serve not only to delineate concepts but also to invite exploration, foster conceptual flexibility, and reinforce the logic of mathematical discourse. As such, effective pedagogy must emphasize dynamic engagement over passive reception, enabling students to internalize definitions as evolving, functional components of their reasoning toolkit.

2.3 The conceptual definition model (CDM): An instructional framework for deep definitional fluency

The Conceptual Definition Model (CDM) reconceptualizes definitions not as static linguistic artifacts, but as dynamic logical frameworks that support deep mathematical reasoning. Central to this model is the operationalization of the biconditional "if and only if" structure, which serves as a pedagogical engine for exploring and internalizing definitions.

CDM guides learners through a structured process of recursive reasoning, example-based validation, edge-case analysis, and logical reconstruction. By doing so, it moves students beyond rote memorization and encourages fluency that is both conceptual and adaptable.

The model incorporates several key elements:

- Logical decomposition: Breaking down definitions into necessary and sufficient conditions.
- Boundary testing: Using counterexamples and edge cases to define conceptual limits.
- Example and non-example construction: Building a spectrum of representative and near-miss cases to solidify meaning.
- Recursive framing: Connecting new definitions to foundational ones through inductive reasoning.

By integrating these strategies, CDM fosters a classroom culture where definitions become tools for inquiry rather than endpoints of instruction. It cultivates analytical precision, metacognitive awareness, and mathematical creativity, all of which are essential for advanced mathematical learning and disciplinary engagement (**Figure 1**).

The essential properties are not arbitrary—they reflect a tight logical structure verified through examples and non-examples.

2.3.1 Logical rigor in CDM definitions: The principles of necessity and sufficiency

Definitions in CDM are treated as logically rigorous tools rather than linguistic labels. Their internal validity is grounded in two essential criteria:

- Necessity: Every instance of the concept must satisfy all the defining conditions.
  - Example: A prime number > 1 must have exactly two distinct positive divisors.
  - Counterexample: 1 fails this condition—hence, not prime.

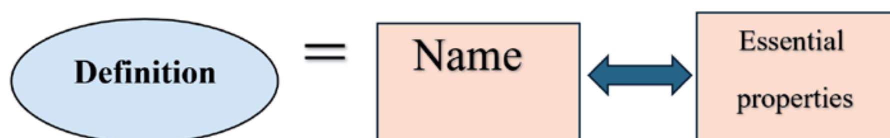

Figure 1.
The logical form of definition in mathematics.

Proposition P	Proposition Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

T: True Proposition, F: False Proposition.

Table 1
Truth table of $P \leftrightarrow Q$.

- Sufficiency: Any object satisfying the conditions must be an instance of the concept.
    - Example: A number with exactly two different positive divisors is prime.
    - Counterexample: 4 has more than two divisors (1, 2, 4).

This dual condition underpins the biconditional truth structure ($P \leftrightarrow Q$), ensuring precision and reliability (**Table 1**).

This dual requirement creates what logicians term a *biconditional* relationship – the "if and only if" structure that forms mathematics' conceptual backbone. The power of this formulation becomes evident when we consider that:

- Proofs depend on definitions as their foundation (e.g., ε-δ continuity proofs)
- Theorems extend definitions' implications (e.g., properties derived from group axioms)
- Generalizations emerge when definitions are modified (e.g., from groups to rings)

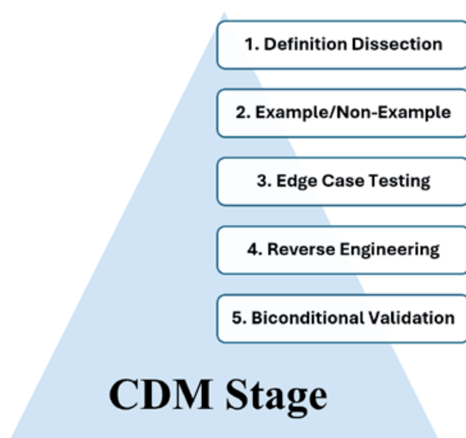

Figure 2.
CDM 5 stages.

CDM Teaching Framework	Detail
1. Definition Dissection	Break apart the definition into:
	• What is the object?
	• What are the necessary properties?
	• Why are these properties sufficient?
2. Example/Non-Example	Students provide:
	• At least two clear examples
	• At least two non-examples
3. Edge Case Testing	Bring in *boundary* or *ambiguous* cases to challenge thinking
4. Reverse Engineering	What Must Be True?
5. Biconditional Validation	If and Only If Validation

Table 2.
CDM stage and detail.

2.3.2 CDM teaching framework: The five-step core

See **Figure 2** and **Table 2**.

2.3.3 CDM teaching framework: The five logical phases

To transform definition learning into a process of structured reasoning, CDM follows five interconnected stages:

1. Definition dissection

   The definition is broken down logically:

   ○ What is the object under study?

   ○ What properties are necessary?

   ○ Why are they sufficient?

   Example:

   "An even number is an integer divisible by 2."

   Logical form: $n \text{ is even} \Leftrightarrow \exists k \in Z \text{ such that } n = 2k$.

2. Example/non-example juxtaposition

   Students actively generate:

   ○ Valid examples (e.g., 4, −8)

   ○ Invalid cases (e.g., 3, 7.5)

This contrast helps define conceptual boundaries by illustrating what the definition *excludes*, not just what it includes.

3. Edge case exploration

Students analyze borderline or ambiguous cases:

- Is 0 even? → Yes, since $0 = 2 \times 0$
- Is -2 even? → Yes, it satisfies the condition.
- Is $\sqrt{2}$ even? → No, because it's not an integer.

These cases deepen understanding and stress-test the logic of the definition.

4. Reverse construction (engineering)
   - Rather than being told a definition, students reverse-engineer it by analyzing commonalities among valid examples.
   - This promotes conceptual ownership and develops metacognitive awareness.

5. If and only if validation
   - Students are prompted to formally test both directions of the definition:
   - Does the condition guarantee the object? (sufficiency)
   - Is the condition required for the object? (*necessity*)

Example (Prime Numbers):

"n is prime ⇔ n has exactly two distinct positive divisors."

Why the CDM Model Matters
The CDM approach reframes definitions as active reasoning tools. It:

- Develops logical precision
- Enhances proof construction
- Fosters deep conceptual understanding
- Supports problem-solving and mathematical communication

Moreover, CDM is inherently adaptable:

- Visual learners engage with diagrams and concept maps

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Step	Purpose
Definition Dissection	Analyze object properties and logic of definition
Example/Non-Example	Contrast valid and invalid cases to sharpen understanding
Edge Case Exploration	Test limits and refine conceptual boundaries
Reverse Construction	Synthesize the definition from shared features of valid cases
If and Only If Validation	Confirm logical sufficiency and necessity

Table 3.
Summary of recursive conceptual steps in function understanding.

- Analytical students thrive on logical validation
- Struggling learners benefit from structured examples and peer discussion [23]

CDM Framework Recap (**Table 3**):

2.3.4 Key takeaways and analytical insights

1. Foundational criterion: Integer domain

    - Parity—the classification of numbers as even or odd—is strictly defined within the domain of integers. Non-integers (e.g., 7.5) are categorically excluded because parity relies on divisibility by 2 without remainder, a property that only integers can possess. This reinforces the importance of respecting definitional boundaries in mathematics.

2. Parity and negativity: Sign irrelevance

    - The property of evenness is invariant under sign. That is, negative numbers like −8 are still even, as they fulfill the same condition (divisible by 2 without remainder). This is consistent with modular arithmetic, where parity is based on congruence modulo 2: if $x \equiv 0 \ (mod\ 2)$, then x is even, regardless of sign.

3. Definitional precision over intuition

    - Misclassifying numbers often stems from intuitive reasoning rather than definitional rigor. For example, 3 is not odd because it's "small"—it fails because it leaves a remainder when divided by 2 ($3 \equiv 1$ mod 2). This binary nature of definitions—either fully satisfied or not—underscores the categorical sharpness of mathematical logic.

4. Pedagogical significance

    - Students frequently misinterpret parity when they overgeneralize the concept (e.g., assuming decimals can be even). Such misconceptions can be prevented through explicit instruction on the scope of definitions, such as clarifying that "even numbers = elements of 2Z." Reinforcing domain-specific boundaries builds foundational accuracy.

5. Abstract algebraic generalization

   - From an advanced standpoint, parity corresponds to the partitioning of Z into two equivalence classes (mod 2), forming the ideal 2Z and its coset 2Z + 1. This connects basic arithmetic properties with deeper structures in abstract algebra, providing a bridge between elementary concepts and higher-level theory.

2.3.5 Model overview

The Conceptual Definition Model (CDM) unfolds through five interrelated phases, each designed to deepen learners' understanding of mathematical and scientific concepts by refining the way definitions are constructed, tested, and internalized:

1. Definition dissection

   In this initial phase, learners break down a given concept to identify its necessary and sufficient conditions. For instance, when defining a triangle, essential properties such as having three sides and the internal angles summing to 180° are distinguished and examined for their role in constituting the concept.

2. Example vs. non-example comparison

   Conceptual clarity is enhanced by juxtaposing correct examples with near misses—cases that closely resemble the concept but fail to meet one or more critical properties. For example, while a square qualifies as a rectangle, a rhombus without right angles does not.

3. Boundary case analysis

   This reflective phase probes the robustness of the definition by applying it to edge cases or ambiguous scenarios. Does a triangle with two sides of zero length still meet the definition? Such explorations refine the precision and scope of the concept, exposing hidden assumptions or oversights.

4. Reverse construction

   In this generative step, students reconstruct the definition by abstracting from multiple valid examples, ensuring the resulting formulation maintains logical coherence and completeness. This reverse engineering approach solidifies understanding through synthesis rather than memorization.

5. Biconditional validation ("If and Only If")

   Finally, the model demands a rigorous test of the definition's biconditional accuracy: a concept must include all and only those cases that meet its conditions. For example, the statement "A figure is a square if and only if it is a rectangle with four equal sides" is examined to confirm that it fully encapsulates the concept without omission or redundancy.

2.4 Discussion on instructional strategies for teaching mathematical definitions

The instructional strategies outlined in this section offer a comprehensive, theory-informed approach to teaching mathematical definitions in a way that promotes conceptual understanding, critical thinking, and student engagement. By aligning each strategy with phases of the Conceptual Development Model (CDM), this framework provides a coherent roadmap for fostering deep learning through structured exploration, analysis, reflection, and application.

At the heart of these strategies lies the constructivist belief that students build knowledge most effectively when they actively engage with content. Constructivist Definition-Building and Contrastive Example Analysis empower learners to interact directly with examples and non-examples, fostering an internalization of necessary and sufficient conditions. These methods promote a shift from rote memorization to genuine understanding, in which students generate, test, and refine their own definitions.

This aligns with Arthur and Dagan's [17] findings that definitional flexibility significantly improves problem-solving transfer in advanced mathematics. Moreover, strategies such as Definitional Stress Testing and Role-Playing introduce students to the nuanced and often ambiguous nature of mathematical definitions. By engaging with edge cases and defending or challenging definitions in structured debates, students develop higher-order thinking skills and an appreciation for the logical precision inherent in mathematics.

The inclusion of Visual Reconstructive Mapping and Multimodal Definition Journaling acknowledges the importance of multiple representations in supporting diverse learners. These strategies tap into visual, linguistic, and reflective modalities, aligning with cognitive theories such as Dual Coding and Vygotsky's Sociocultural Theory. They also promote metacognitive awareness by encouraging learners to revisit and revise their understanding over time.

Further, strategies like Definitional Biographies and Recursive Definition Unpacking highlight the evolving and recursive nature of mathematical thought. These approaches allow students to see definitions not as static rules but as dynamic constructs shaped by historical, logical, and conceptual developments.

Collectively, these strategies not only support students in mastering formal mathematical definitions but also cultivate transferable skills such as argumentation, reasoning, and self-assessment. The implementation framework, which links each strategy to a CDM phase, estimated duration, and a suggested assessment tool, ensure that educators can adapt these approaches flexibly across various instructional contexts.

Ultimately, redefining the way definitions are taught in mathematics classrooms is crucial for nurturing mathematically literate learners. This set of strategies invites educators to move beyond transmission-based instruction and embrace practices that are student-centered, cognitively rigorous, and grounded in both theory and empirical evidence. When implemented thoughtfully, these methods have the potential to transform definitions from isolated statements into powerful tools for inquiry, communication, and meaning-making in mathematics.

3. Conclusion and future directions

To educate a generation of thinkers—not mere memorizers—we must radically rethink how we approach definitions in mathematics. Definitions are not static

artifacts to be stored in memory; they are mental architectures—dynamic, logical blueprints that shape how students perceive, reason, and construct meaning.

This alignment with Bloom's taxonomy of educational objectives—particularly in developing analysis and synthesis levels *Bloom [24],* demonstrates CDM's pedagogical robustness. Rather than trapping learners in rigid formalism or unstructured exploration, the CDM offers a balanced scaffold—combining structure and discovery, precision and agency.

By anchoring instruction in clear properties, powerful examples, rich non-examples, and reflective engagement, definitions become instruments of insight. A well-understood definition is not a limit but a lens, not a wall but a window. It guides students not only in knowing what something is, but in understanding why it is that way—and what it could become under different conditions.

Ultimately, when students internalize that a mathematical definition is a model of reality, a language of abstraction, and a promise of consistency, we are no longer teaching math alone. We are teaching how to think, how to reason, and how to learn for life [25].

"If students master definitions, they master mathematics. But more importantly, they master the foundations of disciplined thought."

4. Future directions and recommendations

Given the powerful cognitive potential of the CDM, the following avenues are essential for future exploration:

- Cross-disciplinary research:

- Evaluate the CDM's impact across diverse mathematical domains—such as calculus, discrete math, set theory, linear algebra, and formal logic—to determine its universality and adaptability.

- Comparative pedagogical studies:

- Contrast CDM with other teaching models (e.g., inquiry-based learning, direct instruction, and flipped classrooms) using rigorous experimental and quasi-experimental designs.

- Differentiated application:

- Investigate how the CDM can be tailored for various student groups—including gifted learners, neurodivergent students, and those with learning difficulties.

- Teacher training and curriculum design:

- Promote the integration of CDM into teacher education programs, curriculum guidelines, and assessment rubrics to ensure that definition instruction becomes a central pillar of conceptual teaching.

- Classroom culture shift:

- Encourage educators to foster classrooms where definitions are questioned, debated, visualized, and reconstructed, not just copied and recited [26].

The definition is often the first thing a student sees—but it should never be the last thing they understand. By embedding the principles of the CDM in our teaching, we not only clarify what mathematics is—we reveal what it means to think mathematically.

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Chapter 4

Bridging Complexity in Mathematics Education: Integrating Hard and Soft Theories through Substructure and Superstructure

Maurice I. Yolles

Abstract

This chapter examines how the metacybernetic complexity paradigm applies to mathematics education. It explores the interplay between the complexity continuum and the substructure-superstructure distinction to analyse the balance between structured, deterministic models and flexible, adaptive approaches in curriculum design and pedagogy. Rather than strictly aligning hard and soft theories with substructure and superstructure, the chapter demonstrates how both structured (hard) and flexible (soft) approaches operate within and across these realms. Hard theories, typically associated with procedural fluency, standardised assessments, and formal knowledge acquisition, tend to align more with the superstructure due to their tangible, externalised nature. Soft theories, which emphasise conceptual understanding, student agency, and socio-emotional engagement, are more characteristic of substructure due to their intangible and internalised qualities. However, substructure and superstructure contain gradations along the complexity continuum, allowing for the coupling of hard and soft elements within educational systems. A focus lies on challenges such as mathematics anxiety, equity, and student engagement. The chapter argues that rigid, exclusionary structures and overly fluid, incoherent pedagogies can undermine effective learning. To address this, an adaptive, recursive framework is proposed that dynamically integrates structured and flexible elements to support cognitive improvement. Extending to curriculum design, teaching methodologies, and policy-making, the discussion explores strategies to maintain structured instruction while fostering reflective inquiry, collaboration, and culturally responsive teaching. Synthesising theoretical insights with pragmatic approaches, the chapter outlines a mathematics education paradigm that balances measurable learning outcomes with human-centred principles. This ensures that students develop mathematical competence alongside resilience, critical thinking, and lifelong engagement with learning. While hard theories often manifest visibly through superstructural mechanisms such as curricula and assessments, structured cognitive processes can also be regarded as substructural hard elements. Similarly, while soft theories frequently operate at an affective, substructural level, they also inform flexible pedagogical practices in

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the superstructure. This reflects the continuum-based view adopted in this chapter, where hard and soft elements recur within and across both structural realms.

Keywords: metacybernetic paradigm, complexity continuum, substructure-superstructure, adaptive learning, hard and soft theories, conceptual understanding, student agency, mathematics anxiety, holistic development, culturally responsive teaching

1. Introduction

This chapter explores the role of complexity, the interplay of hard and soft theories along a continuum, and the substructure-superstructure dynamics in mathematics education by integrating insights from metacybernetics [1]. Consideration of the complexity continuum [2] provides a theoretical framework for understanding how structured (hard) and flexible (soft) approaches operate within and across both substructural (intangible) and superstructural (tangible) dimensions [3]. These conceptualisations accommodate internal gradations of complexity along the hard-soft spectrum. Substructure encompasses internal processes like cognitive models, emotions, and belief systems, ranging from highly subjective (soft) conceptual understanding to more structured thought patterns and formal reasoning (hard). Superstructure involves external systems such as curriculum design, institutional policies, and teaching methodologies, varying from flexible norms (soft) to rigid laws and standardised curricula (hard). This perspective is crucial in mathematics education, as overemphasising rigid curricula and assessment-driven methods can hinder deeper conceptual understanding and student engagement. By recognising the relationship between these dimensions, an adaptive framework can be developed that integrates structured instruction with flexible, inquiry-based approaches.

Yolles and Fink [3] conceptualise substructure and superstructure as distinct while connected within a given context, aligning with metacybernetics. In complex systems, substructure pertains to intangible elements like cognitive processes and belief structures, while superstructure includes tangible components such as curriculum design and teaching methodologies. Metacybernetics acknowledges that substructure and superstructure are not isolated but are dynamically interconnected realms within a complex system. These realms form a continuum of structures and processes that interact and influence one another, challenging simplistic binaries and emphasising a relational ontology.

In mathematics education, the duality of understanding the relationship between hard and soft elements is essential. Mathematics consists of hard structures, such as formal rules and algorithms, and soft structures, including heuristics and problem-solving strategies. Overemphasising rigid procedural instruction can lead to mechanical proficiency without conceptual understanding, while too much reliance on exploratory approaches may result in misconceptions. Thus, a balance between hard and soft elements supports procedural fluency and conceptual depth.

Advanced mathematical reasoning involves rigidity. It also includes creativity, abstraction, and adaptability, qualities associated with soft elements. For instance, proof-based reasoning requires logical structures (hard) and intuition in forming conjectures (soft). Understanding the connection between these elements is important to accommodate diverse learning styles and encourage mathematical creativity. Some students excel in structured, procedural learning, while others thrive in visual

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and exploratory approaches [4]. Effective mathematics education should integrate to support various learners. For instance, geometric proofs can be formalised algebraically (hard) or reasoned through visual intuition (soft), with both enhancing understanding.

Balancing hard and soft elements is essential for developing problem-solving skills and mathematical creativity. Hard elements provide rules and structure, while soft elements allow for open-ended exploration and adaptive reasoning. Creativity in mathematics involves restructuring problems, finding alternative solutions, and making connections—processes that blend hard and soft thinking [5]. In combinatorics, solutions may rely on formulaic counting rules (hard) or intuitive reasoning, using symmetry and patterns (soft). This interconnection is important during the transition from school mathematics to higher-level mathematics. Early education often emphasises rigid procedures, while advanced mathematics requires abstract reasoning and generalisation [6]. Students accustomed only to procedural approaches may struggle with deeper conceptual notions like limits and continuity.

Finally, considering hard and soft elements informs pedagogical approaches and curriculum design in mathematics education. The hard-soft continuum operates within the concepts of substructure and superstructure, represented in **Figure 1** for illustrative purposes. Each realm exhibits internal gradations, transitioning from highly structured (hard) elements to more flexible (soft) ones as complexity increases. Rather than existing in simple opposition, substructure and superstructure co-evolve, dynamically adjusting to this increasing complexity. The graph illustrates that both realms gradually lose rigid structures as they incorporate adaptive elements. In mathematics learning, for instance, the superstructure begins with procedural foundations—such as rules for matrix multiplication—and evolves to include flexible practices that connect these procedures to real-world applications. Concurrently, the substructure shifts from structured cognitive frameworks (hard) to more open-ended, affective processes (soft). This dynamic process is central to learning, particularly in mathematical problem-solving, where students engage in a continuum of adaptation, toggling between rigid extremes and progressing from concrete

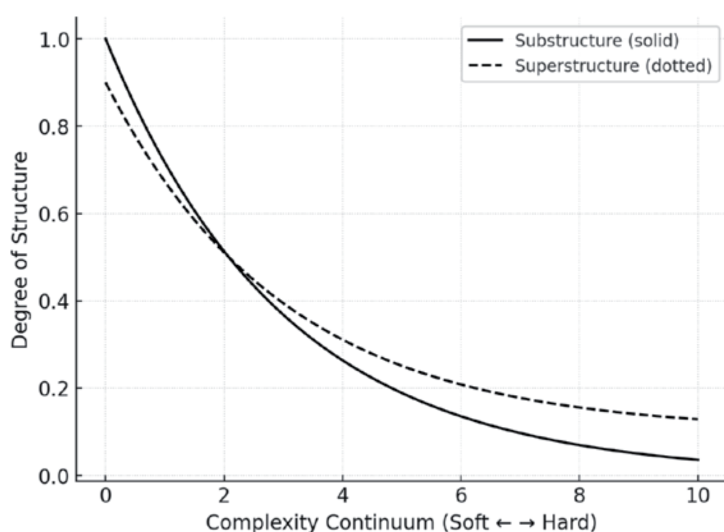

Figure 1.
Graduations of hard and soft elements in substructure and superstructure.

procedures to abstract reasoning. Effective feedback plays a crucial role in this process by enabling students to reflect on their understanding and adjust their approaches. This interplay of structured cognitive patterns and interpretative, motivational aspects fosters deeper engagement with mathematical content.

Therefore, the concave decline depicted in the graph reflects how hard and soft elements are interconnected, ultimately shaping students' educational experiences. **Figure 1** shows that while substructure and superstructure, as defined by Yolles [1] and further elaborated by Yolles and Fink [3], are distinct realms, they are fluid and characterised by internal gradations rather than fixed extremes. Crucially, the interplay between these realms is mediated by feedback mechanisms. The distinction between substructure, the "not directly observable" realm encompassing internal processes and cognitive structures, and superstructure, the "directly observable" realm representing tangible outcomes and behaviours, underscores their core components. For Hattie and Timperley [7], effective feedback is important to bridging these realms, allowing students to adjust their internal processes based on external observations and vice versa. Feedback mechanisms are essential for the dynamic substructure-superstructure interaction, enabling learning development. They provide the necessary bridge for information flow and influence between these realms, ensuring an effective response to internal and external changes. For example, feedback on student performance (superstructure) not only alters classroom instruction but also conditions belief systems and motivational frameworks (substructure), which in turn affect future engagement and learning trajectories—closing the loop.

Building on this foundation, this chapter explores how these distinctions manifest in mathematics education, particularly concerning complexity. It examines the role of hard and soft theories, the substructure-superstructure dynamic, and the integration of cognitive, affective, sociocultural, and spiritual dimensions. It analyses how structured (hard) and flexible (soft) pedagogical models operate across substructural (intangible) and superstructural (tangible) dimensions. These conceptualisations allow for a gradational view of complexity, recognising that both substructure and superstructure encompass a continuum of rigidity and adaptability.

The aim is to develop a complexity-informed framework for mathematics education that balances structured instruction with flexible, inquiry-based approaches. Traditional mathematics education often prioritises rigid curricula and formal reasoning at the expense of deeper conceptual understanding, creativity, and student engagement. By acknowledging the dynamic connection between substructure and superstructure, this chapter advocates for an adaptive pedagogical model that accommodates cognitive processes, emotional engagement, sociocultural influences, and broader ideational values. This perspective is important in addressing mathematics learning anxiety, which arises from interactions between rigid instructional models, assessment pressures, and students' cognitive and affective experiences [8]. Recognising the fluidity of hard and soft elements within educational structures allows for designing intellectually rigorous and personally meaningful learning environments.

This conceptualisation of substructure and superstructure extends beyond a simple binary classification. Complexity theory reveals a continuum of interrelated elements, where both dimensions interact dynamically. The remainder of the chapter is structured as follows. Section 2 establishes the theoretical foundation by exploring the principles of complexity in education, situating mathematics learning within a dynamic, evolving system. It highlights how complex systems theory, self-organisation, and adaptive learning processes inform contemporary educational thought. Additionally, it examines mathematics learning anxiety as an emergent phenomenon

shaped by cognitive, affective, and sociocultural interactions. This section considers how rigid instructional models, assessment pressures, and negative past experiences contribute to anxiety and how student-centred approaches can mitigate its effects. Section 3 discusses the challenges of mathematics education through a complexity lens, addressing cognitive overload, rigid curricula, and the underrepresentation of affective and sociocultural factors in traditional pedagogical models. The next section introduces the integration of substructure and superstructure in pedagogical strategies, demonstrating how abstract mathematical principles (substructure) can be meaningfully connected with students' lived experiences and broader educational contexts (superstructure). Building on this foundation, Section 5 presents a comprehensive framework for mathematics education that synthesises cognitive, affective, sociocultural, and spiritual dimensions, offering a balanced model of structure and adaptability. Section 6 provides a case study illustrating the practical application of this framework, showcasing how integrating substructural and superstructural elements in instructional design can enhance student engagement and conceptual understanding. The subsequent section explores the broader implications of this approach, considering its potential to inform curriculum development, teacher training, and educational policy. Finally, Section 8 synthesises the key insights from the chapter, reflecting on the transformative potential of this holistic approach in reshaping mathematics education.

2. Foundations of complexity in education

The foundations of complexity in education are rooted in the interplay between hard and soft theories, which operate as substructures and superstructures. This duality provides a comprehensive view of how educational systems function and adapt. Hard theories, characterised by their measurable, functional, and operational components, provide stability and efficiency, ensuring educational practices are grounded in clear, quantifiable goals [9]. In mathematics education, they manifest through curriculum standards, performance metrics, and structured problem-solving frameworks, enabling consistency and effective instruction.

Soft theories, by contrast, emphasise human-centred, interpretive, and ideational dimensions that guide the adaptability and purpose of educational systems. These theories prioritise students' emotional and cognitive well-being, fostering engagement, addressing mathematics anxiety, and promoting growth mindsets [10]. Yolles [2] further situates soft theories within the human condition, recognising that they can be blended with hard theories to create more harmonious educational systems. Soft theories thus highlight the importance of intangible substructural values—such as emotional engagement and equity—that inform tangible superstructural elements, including curricula, assessment frameworks, and pedagogical strategies. The interconnection between hard and soft theories is essential for the holistic development of educational systems. Hard theories provide a necessary structure for consistency and reliability in instruction, while soft theories ensure that learning remains meaningful, responsive, and engaging for students. Recognising this balance allows educators to develop adaptive teaching approaches that integrate stability and flexibility, fostering students' growth across multiple dimensions [9].

Soft theories are particularly valuable in mathematics education by addressing the affective and cognitive needs of the student. For instance, fostering engagement involves creating stimulating and interactive learning environments, making mathematics more relevant and enjoyable [10]. Additionally, managing mathematics

anxiety is a crucial application of soft theories. Research has shown that anxiety can significantly impact students' performance and attitudes towards the subject [11]. By incorporating strategies such as mindfulness exercises, positive reinforcement, and supportive teacher-student relationships, educators can create a more inclusive and positive learning experience. This dynamic relationship between hard and soft theories underscores the importance of an integrated framework in mathematics education. It balances structure with adaptability, ensuring rigorous learning outcomes and a supportive, student-centred environment.

In their dynamic relationship, hard and soft theories connect to create a balanced and effective educational system. Measurable functional components of hard theories provide the structure for consistent and reliable instruction. At the same time, the human-centred, interpretive aspects of soft theories ensure that education remains a meaningful and engaging process for all involved. Understanding and using the connection between hard and soft theories can help educators develop a more subtle and adaptive approach to teaching and learning. This holistic perspective allows for the creation of educational systems that are both stable and flexible, capable of meeting the diverse needs of students and fostering their growth in multiple dimensions.

Table 1 illustrates how substructure and superstructure interact with the complexity continuum, highlighting the gradation between hard and soft approaches within each realm. It allows for improved recognition of how hard and soft theories function within mathematics education, ensuring a balanced approach that fosters procedural mastery and deep conceptual engagement. Reference to the implicate and explicate in the table argues for the substructure and superstructure being fundamentally interconnected, where the hidden order of the implicate directly shapes and conditions the manifest reality of the explicate.

This perspective is important in mathematics education. Traditional "hard" theories often emphasise structured, teacher-centred instruction that transmits specific knowledge and skills. While this approach provides a clear framework for curriculum design, it may not fully address the complexities of individual learning processes or the influential cultural contexts involved. Critics argue that

Realm	Hard (Structured, Deterministic)	Hybrid (Structured Flexibility)	Soft (Flexible, Adaptive)
Substructure (Internal, Implicate)	Formal logical reasoning. Algorithmic problem-solving. Rule-based cognition	Metacognitive reflection *with* structured prompts. Conceptual exploration *guided by* frameworks (e.g., Polya's problem-solving steps)	Intuitive understanding. Open-ended reflection. Emergent creativity
Superstructure (External, Explicate)	Standardised curricula. Rigid assessments (e.g., timed exams). Direct instruction	Inquiry-based learning *with scaffolds* (e.g., guided questions, rubrics). Student agency *within* clear goals (e.g., mastery tracks). Culturally responsive teaching *aligned to* standards	Pure discovery learning. Unstructured play-based math. Student-designed projects

Table 1.
Hard-soft continuum across substructure and superstructure.

overemphasising rigid methodologies can lead to a narrow understanding of mathematics, potentially disengaging students who benefit from more exploratory and contextually relevant learning experiences [12]. In contrast, "soft" theories promote a more flexible, student-centred approach that recognises the emergent and dynamic nature of learning. These theories draw on sociocultural perspectives, emphasising that learning occurs through meaningful social interactions and cultural practices. For example, Vygotsky's [13] theory highlights the role of social context and cultural tools in cognitive development [14]. Integrating "soft" theories into mathematics education involves acknowledging the socio-affective dimensions of learning, including emotions, attitudes, and beliefs, which significantly impact student engagement and success. Research indicates that addressing these factors enhances inclusivity and effectiveness in educational practices [15]. Thus, a balanced integration of both "hard" and "soft" theoretical perspectives is essential.

Traditional teaching methods often prioritise rigid, procedural methodologies aligned with "hard" theories—characterised by deterministic structures and well-defined parameters. Such approaches are effective for inculcating specific skills, but they often neglect the complexity and variability inherent in the learning process. In contrast, contemporary challenges in mathematics education necessitate a subtle perspective incorporating "soft" theories, which recognise adaptability and the socio-affective dimensions of learning. This is supported by Davis and Sumara [9], who explore the implications of complexity in education, asserting that rigid pedagogical frameworks may fail to engage students meaningfully. They advocate for an educational approach that embraces the unpredictability of learning environments and considers the emotional and social contexts that shape student experiences. By acknowledging these complexities, educators can foster more inclusive and effective learning environments. Thus, a shift towards methodologies prioritising flexibility and student-centredness is essential in addressing the diverse needs of learners in mathematics.

The complexity of mathematics education arises from the diversity of learner cognitive abilities and the sociocultural contexts in which they learn. Systemic approaches have been proposed to address the multifaceted nature of teaching and learning mathematics, emphasising the need for adaptable and context-sensitive pedagogical methods [16]. The interconnection between the tangible superstructure and the intangible substructure offers a valuable framework for understanding educational paradigms. The superstructure shapes the cultural and institutional context of learning, encompassing observable systems and structures such as curriculum, teaching methods, and learning policies. These reflect broader ideologies and societal expectations [17–19]. The substructure influences how individuals think, learn, and interact, encompassing mindset, mental models, and motivation. It includes the internal cognitive and affective processes that mediate how learners experience and respond to the external structures presented by the superstructure.

This framework suggests that the rigidity of "hard" theories within the substructure, such as prescriptive methodologies and assessment-driven pedagogies, can conflict with the evolving nature of superstructural values, which may demand more inclusive and adaptive teaching approaches. This tension is particularly evident in mathematics education, where traditional frameworks often prioritise rigidity and quantification of learning, potentially impeding the more adaptive, student-centred values that contemporary educational paradigms endorse. Hard theories typically emphasise teacher-centred instruction and standardised curricula focused on measurable outcomes, which can stifle creativity and inter-student engagement. Conversely, educational systems increasingly value the inclusion of sociocultural

factors, emotional engagement, and adaptive learning models, aligning more closely with "soft" theories [14, 20]. Recognising the dynamic tension between the tangible superstructure and intangible substructure allows for a deeper understanding of the complexities of mathematics education, progressing the need for a balanced pedagogical model that combines structured, evidence-based methodologies with a flexible, student-centred approach that responds to learners' evolving social, cultural, and cognitive needs.

Cognitive factors are relevant to the mental processes involved in mathematics learning, including problem-solving, reasoning, and memory. Models focused on mathematical motivation and engagement [21] offer insights into how cognition impacts understanding. These models highlight the importance of developing a deep, conceptual understanding of mathematical principles to enhance student engagement and motivation. Affective factors involve emotions, attitudes, and beliefs students hold towards mathematics. Research by Goldin et al. [22] emphasises the role of these dimensions, indicating that emotional responses can significantly impact engagement and achievement in mathematics. Addressing affective factors is crucial for creating a supportive learning environment that fosters positive attitudes and emotional connections with mathematics. By promoting a climate of emotional support, educators can reduce anxiety and build confidence, ultimately enhancing student engagement and perseverance. Sociocultural factors encompass the contexts in which students learn mathematics. Vygotsky's sociocultural theory underscores the significance of social interactions and cultural tools in cognitive development, reinforcing the need for teaching methods responsive to culturally diverse backgrounds [23]. Educators can create a more inclusive learning environment by integrating culturally relevant strategies and recognising the collective knowledge students can bring from their communities. In addition to cognitive, affective, and sociocultural dimensions, the spiritual aspect of mathematics education includes the underlying ideational values and principles that shape teaching and learning. Drawing from Yolles [24], "spirit" refers to the cultural frameworks informing educational practice and influencing how knowledge is valued. In mathematics education, this spiritual dimension encourages curiosity, creativity, and deeper understanding, positioning mathematics as both a technical discipline and a transformative process that supports intellectual development. This perspective enriches student learning experiences by fostering intrinsic motivation and reinforcing the broader relevance of mathematics as a tool for problem-solving and meaningful inquiry.

3. Complex challenges in mathematics education

Mathematics education faces significant challenges due to the interaction between the educational substructure and the superstructure [25]. The complexity of mathematics education arises from the need to balance these two components, ensuring that the external structures support and align with the internal, dynamic learning processes. At the substructural level, students' cognitive abilities, emotional responses, and attitudes towards mathematics shape their learning experiences. Cognition, problem-solving, logical reasoning, and memory interact with emotional aspects such as anxiety, confidence, and motivation, all impacting student engagement with mathematics. For instance, students struggling with mathematics anxiety may find it difficult to focus on problem-solving tasks, leading to disengagement and poor performance. These internal affective challenges require that educators adopt

strategies that cater to emotional well-being, fostering a positive mindset and motivation towards learning.

The substructure relevant to mathematics learning depends, obviously, on cognitive and emotional factors. According to Dweck's work on mindset theory [26], students' beliefs about their abilities significantly affect their learning outcomes. Those with a growth mindset—the belief that abilities can be developed through effort—are more likely to embrace challenges and persist with difficulty. In contrast, students with a fixed mindset—believing that intelligence is static—tend to avoid challenges and may give up more easily. This aligns with student emotional responses towards mathematics: a growth mindset fosters resilience and perseverance, while a fixed mindset can lead to disengagement, particularly in subjects perceived as difficult or intimidating, such as mathematics. Furthermore, emotional factors like mathematics anxiety can prevent students from attempting to solve problems, thus exacerbating the gap between their cognitive potential and their actual performance [27].

On the superstructural side, the external elements of mathematics education—such as standardised curricula, rigid assessment models, and teacher-centred instructional approaches often fail to address the fluidity and complexity of individual learning experiences. These structures are necessary for creating consistency and measurable outcomes but can conflict with student needs. Rigid, procedural teaching methods can inadvertently stifle creativity and fail to accommodate diverse learning styles. For instance, standardised testing, though useful for measuring certain aspects of performance, may not reflect the depth of understanding or problem-solving skills that students develop over time. As a result, these superstructural elements may struggle to meet the needs of students whose cognitive and emotional processes do not align with standardised expectations [28].

The tension between the substructure and superstructure in mathematics education becomes evident when the inflexibility of external structures clashes with the fluid and adaptive nature of internal processes. Traditional educational models are often grounded in hard theories, emphasising rigid, measurable frameworks, like standardised testing and prescriptive curricula. These models typically neglect the variability and unpredictability inherent in student learning, ignoring the affective and cognitive dimensions that influence how students experience mathematics. Such rigid systems may exclude or disengage students who require a more flexible, supportive learning environment. This is particularly evident in standardised assessments prioritising procedural knowledge over conceptual understanding. This divide often leaves students with weak conceptual foundations struggling to make meaningful connections in more advanced topics.

Research indicates that traditional assessment methods in mathematics primarily involve standardised tests that assess students' knowledge of facts, formulas, and procedures. This approach often results in a superficial understanding of mathematics and a lack of flexibility in tackling real-world challenges and more advanced problem-solving [29]. Furthermore, studies have found that a large group of students may score high in procedural tasks but low in conceptual ones, suggesting that their grades reflect performance on procedural knowledge rather than a deep understanding of underlying concepts [29]. Additionally, the affective dimensions of learning, such as mathematics anxiety, can significantly impact students' engagement and performance. Zafra [30] has recognised that many students experience nervousness and worry about mathematics, which can affect their performance and future study choices. These findings underscore the need for educational models that balance procedural proficiency with conceptual understanding and address both the cognitive and affective dimensions of learning.

3.1 The role of mindset in mathematics learning

Having referred to Dweck's [26] mindset theory, there is a need for a caveat here. Her conceptualisation has been useful in shaping educational discourse, particularly in understanding cognitive and emotional barriers in mathematics education. Her framework distinguishes between two primary mindsets: fixed and growth. In fixed mindsets, Individuals believe that intelligence is static, leading to avoidance of challenges and fear of failure. In a growth mindset, individuals believe intelligence can be developed, fostering persistence and adaptive learning behaviours. While this model provides insight into motivation and learning persistence, it lacks the complexity to account for broader sociocultural and systemic factors influencing individual agency. Mindset Agency Theory (MAT), developed by Yolles and Fink [3] and extended by Yolles [24], offers a far more comprehensive and dynamic approach. MAT's trait-based model operationalises this complexity by framing mindsets as adaptive systems that interact with both structured (hard) and flexible (soft) learning environments, as illustrated in the hard/soft continuum (**Table 1**).

MAT is derived from the metacybernetic paradigm, which considers the interplay between substructural (internal cognitive-affective processes) and superstructural (external sociocultural influences) elements. Unlike Dweck's dichotomous model, MAT employs a trait-based structure involving 21 trait variables to model agency character as a complex, adaptive system. MAT acknowledges that mindset is not a static personality trait but evolves dynamically through internal and external forces.

Yolles and Fink [3] integrate cognition and affect within MAT, while Yolles [24] extends this model to include spirit, embedding individual agency within a broader sociocultural system. This allows for sophisticated modelling of the dynamic interactions between students' cognitive, emotional, and social realities. The key differences between the two theories can be summarised in **Table 2**.

Yolles [24] categorises mindsets based on cognitive, affective, and spirit orientations. These may be related to Dweck's growth vs. fixed mindset framework, where MAT mindsets may be seen as primarily growth-oriented or fixed-oriented based on openness to adaptation, learning, and self-transcendence. Thus, growth mindsets emphasise development, self-expansion, adaptability, and openness to change. These mindsets reflect learning-oriented, resilient, and self-evolving

Attribute	Dweck's mindset theory	Mindset Agency Theory (MAT)
Nature of mindset	Fixed vs. growth (binary)	Dynamic, multi-trait model
Focus	Individual beliefs about intelligence	Cognitive, affective, and sociocultural interactions
Change mechanism	Encouraging growth mindset through feedback	Complex adaptive system with recursive feedback loops
Influences	Personal experiences and feedback	Internal traits, cultural norms, and institutional structures
Flexibility	Static categories	Configurable and evolving mindsets
Agency	Limited to motivation and persistence	Holistic interaction of cognition, affective emotion, spirit and society

Table 2.
Comparative overview of Dweck's theory vs. MAT.

Affect mindsets	Affect traits	Cognition mindsets	Cognition traits	Spirit mindsets	Spirit traits	Growth vs. fixed
Stimulation-oriented	Individualism-oriented	Dispositional mindsets	Sociocultural traits	Transcendent individualism (TI)	Self-actualisation, self-transcendence	Growth
Dominant Sanguine (DS)	Stimulation, ambition, dominance	Hierarchical individualism (HI)	Intellectual autonomy, mastery + affective autonomy	Spirited leadership (SL)	Inspirational influence, willpower, assertive creativity	Growth (if adaptive)/fixed (if dominance-focused)
Moderate Sanguine (MD)	Stimulation, ambition, submission	Egalitarian individualism (EI)	Intellectual autonomy, mastery + affective autonomy	Innovative expression (IE)	Adaptive ideation, self-realisation, self-expansion	Growth
Reformer Melancholic (RM)	Stimulation, protection, dominance	Hierarchical synergism (HS)	Intellectual autonomy, harmony, hierarchy	Spiritual governance (SG)	Wisdom, structured ethical engagement, systemic vision	Growth (if wisdom-driven)/fixed (if control-driven)
Subversive Melancholic (SM)	Stimulation, protection, submission	Egalitarian synergism (ES)	Intellectual autonomy, harmony, egalitarianism	Harmonic compassion (HC)	Empathetic resonance, inner peace, relational synchrony	Growth
Containment-oriented	Collectivist-oriented	Sociocultural mindsets	Ideational traits	Transcendent collectivism (TC)	Communal actualisation, shared purpose, ethical coherence	Growth
Expansive choleric (EC)	Containment, ambition, dominance	Hierarchical populism (HP)	Embeddedness, mastery + affective autonomy	Directive mobilisation (DM)	Visionary strategy, collective drive, organisational cohesion	Growth (if adaptive leadership)/fixed (if authoritarian)
Compliant phlegmatic (CP)	Containment, ambition, submission	Egalitarian populism (EP)	Embeddedness, mastery + affective autonomy	Cooperative flourishing (CF)	Mutual empowerment, collective creativity, adaptive cooperation	Growth
Defensive choleric (DC)	Containment, protection, dominance	Hierarchical collectivism (HC)	Embeddedness, harmony, hierarchy	Guardian ethos (GE)	Protective stewardship, ethical leadership, responsibility	Fixed (if defensive)/growth (if ethical stewardship)
Dormant phlegmatic (DP)	Containment, protection, submission	Egalitarian collectivism (EC)	Embeddedness, harmony, egalitarianism	Altruistic synergy (AS)	Community bonding, relational integrity, compassionate action	Growth

Table 3.
Comparing MAT mindsets with Dweck's growth vs. fixed mindsets.

cognitive-affective-spirit dynamics. Fixed Mindsets emphasise stability, rigid hierarchies, resistance to change, and protective mechanisms. These mindsets reflect conserving, hierarchical, and constraint-based cognitive-affective-spirit dynamics. A comparison on this basis is made in **Table 3**.

Key observations on growth versus fixed mindsets reveal that individualism-oriented mindsets can foster or hinder personal development depending on whether they lean towards growth or fixity. Growth-oriented traits like TI, IE, and HC focus on self-expansion, innovation, and learning. These mindsets encourage adaptability and self-awareness, allowing individuals to evolve through challenges. In contrast, fixed mindsets, exemplified by traits like DS and RM, may become entrenched if power-oriented rather than adaptive, stifling flexibility and innovation.

For collectivism-oriented mindsets, a growth mindset emphasises cooperative self-transcendence and mutual development. Traits like TC, CF, and AS foster shared learning and progress, emphasising collective growth. Conversely, fixed mindsets such as DC and GE may limit collective evolution by fostering defensive and control-driven attitudes that impede mutual development.

Leadership-oriented mindsets also exhibit distinct behaviours under growth and fixed paradigms. Growth-oriented leadership orientations like HI, HP, and HC promote ethical, visionary, and adaptive leadership that drives system-wide transformation. When leadership becomes hierarchical and control-based, as seen in fixed mindsets, it can reinforce dominance structures, stifling the potential for adaptive change and broader organisational growth.

This classification represents Dweck's framework for growth and fixed mindsets within the metacybernetic model by considering how affect-cognition-spirit interactions influence learning and transformation. It helps identify whether mindsets promote growth and adaptability or entrench rigidity and control. Spirit mindsets extend this understanding by integrating transcendence, ethical engagement, and self-actualisation into the cognitive-affective framework. These mindsets reflect higher-order meaning-making, distinguishing TI from TC. Hierarchical spirit mindsets, like SG, GE, and DM, align with leadership structures that prioritise order and control. In contrast, egalitarian spirit mindsets like HC, CF, and AS focus on communal harmony, cooperation, and adaptive empowerment, promoting a more collaborative and inclusive approach to development.

In mathematics education, the integration of MAT offers a nuanced understanding of how cognitive-affective structures evolve. Unlike Dweck's binary model, MAT considers the external sociocultural influences on mindset formation and provides a dynamic framework adaptable to different learning contexts. This approach emphasises the importance of fostering internal motivational factors and recognising institutional and cultural constraints that shape learning experiences. Educational strategies such as inquiry-based learning, problem-based learning, and collaborative activities align well with MAT's holistic framework, allowing students to develop adaptive agency. This contrasts with a simple shift from a fixed to a growth mindset, offering a more complex, responsive approach to student development.

4. Integration of substructure and superstructure in pedagogical strategies

Viewing learning environments as complex adaptive systems allows them to be conceptualised as generic living systems, consistent with Yolles and Fink's [3] framework.

This perspective treats learning systems as dynamic, self-organising entities, similar to living organisms that adapt and evolve in response to internal and external stimuli.

Traditional systems theory might consider the observable system to be defined as a superstructure, while those hidden aspects that influence it are the substructural attributes. This duality is operationalised through recursive feedback loops (e.g., standardised assessments [superstructural] informing differentiated instruction [substructural]), as exemplified in **Table 1**'s hybrid strategies. This perspective can be honed and elaborated on theoretically. Consider the substructural realm, which encompasses the intangible elements that shape the system's ability to adapt, relate, and process information. These elements include affective engagement, emotional responses, and motivational factors directly influencing how students interact with content. Additionally, cognitive flexibility enables students to adjust their thinking and strategies when encountering new information, enabling deeper learning and personal development.

In this living system framework, intelligence is viewed as the capacity for self-organisation. Learning is thus an ongoing, adaptive interaction between students and their environment. This aligns with Piaget's [31] classification of process intelligence into operative intelligence (related to actions and operations) and figurative intelligence (which involves conceptual understanding and the mental representations of those actions). This analogy is further developed respectively through autopoiesis [32] and autogenesis [33]. These concepts enable complex systems to become self-organising and generically living. Operative intelligence aligns with autopoiesis (self-production), which refers to the system's ability to sustain itself through internal processes. Meanwhile, figurative intelligence or autogenesis (self-creation) enables new meanings and mental representations to emerge from the system's internal interactions, facilitating learning.

Framing learning environments as contexts for generic living systems enables the Mindset Agency Theory (MAT) developed by Yolles and Fink [3] to provide a holistic view of education. This perspective emphasises the dynamic and recursive nature of learning, where cognitive, emotional, and motivational factors interact in complex ways, enabling students to self-organise their learning journeys. Just as living systems evolve and adapt to their environment, learners continuously adjust their mental frameworks in response to new challenges and experiences. This perspective moves away from static, linear models of learning, embracing the complexity and adaptability inherent in human development and education.

Returning to substructure and superstructure, practical examples of their interconnective strategies in pedagogy include combining evidence-based practices like differentiated instruction with cultural sensitivity [34]. Differentiated instruction involves tailoring teaching methods and materials to meet the diverse needs of students [35]. When combined with cultural sensitivity, this approach ensures that all students (regardless of their cultural background) can access meaningful and relevant learning experiences.

Another example of the distinction between substructural and superstructural realms can be seen in the use of technology to bridge operational (substructural) and adaptive (superstructural) needs. The substructural realm encompasses internal learning mechanisms like cognitive processing, individual problem-solving strategies, and emotional engagement with educational content. Technology supports these internal processes by enhancing standardised assessments, providing real-time feedback, and offering personalised learning experiences that help students monitor their progress and improve their understanding. The superstructural realm involves the broader

external forces that shape educational systems, including institutional policies, pedagogical frameworks, and societal expectations. Pedagogical innovations, such as integrating adaptive learning technologies or collaborative online platforms, are superstructural because they emerge from and respond to these broader influences. These innovations transform traditional educational models by facilitating new teaching strategies, supporting diverse learning styles, and responding to evolving demands in education and the workforce [36]. Thus, while technology operates at both levels, its role in shaping pedagogical innovation aligns with superstructural adaptation, as it restructures how learning is organised and delivered on a systemic level.

5. A framework for mathematics education

Mathematics education, when examined through the perspective of Mindset Agency Theory [24], can be understood as a complex adaptive system in which cognition, affect, and spirit interact across different levels of organisation. This perspective aligns with the metacybernetic approach, which models systemic agency through the interaction between related substructural and superstructural processes. Applying this framework to mathematics education allows for a deeper understanding of how learning is shaped by cognitive and affective dimensions, as well as by the overarching purpose and values that drive educational engagement.

Cognition, affect and spirit, in mathematics education can be divided into two interrelated levels: superstructure (explicate) and substructure (implicate). The superstructure represents the observable, operational explicate, focusing on problem-solving techniques and formal knowledge acquisition. This aligns with traditional models of mathematics education that emphasise procedural fluency, logical reasoning, and the mastery of symbolic representations. However, within the framework of Mindset Agency Theory, cognition extends beyond operational proficiency to include a substructural implicate, which encompasses intangible, conditioning dimension of cognition (e.g., higher-order thinking, metacognitive awareness, and integrative reasoning), affect (e.g., intrinsic motivation, emotional resilience), and spirit (e.g., cultural values, ethical engagement). Here, students develop metacognitive awareness, allowing them to contextualise mathematical concepts within broader sociocultural frameworks transcending rote mastery.

Affect, or the emotional and motivational dimension of learning similarly manifests at both superstructural and substructural realms. The superstructural explicate, however, involves immediate emotional responses to performance, such as anxiety, confidence, and reactions to feedback loops. These responses, shaped by the deeper implicate order, influence engagement and persistence, shaping students' willingness to engage with mathematical challenges. The substructural implicate pertains to deeper evaluative processes that align affect with long-term goals and personal values, ultimately fostering self-efficacy. These underlying evaluations and values create a foundational affective disposition that conditions how students experience and react to learning. Within MAT, affect is important in sustaining agency, as emotional dispositions (rooted in the implicate) recursively interact with cognitive processes to reinforce or hinder learning trajectories, manifesting as observable emotional responses (in the explicate).

The dimension of spirit, as conceptualised within Mindset Agency Theory and applied to mathematics education, provides an important link between individual agency and broader societal objectives. The superstructural explicate of spirit manifests in the action-oriented intent that underlies curriculum delivery, including pedagogical

choices, instructional design, and institutional priorities. This level ensures that the structural mechanisms of education are functional and purposeful. The substructural implicate, however, encapsulates the overarching purpose of mathematics education, aligning it with societal and cultural values. This realm reflects the ideological and aspirational dimensions of learning—whether mathematics is positioned as a purely technical discipline, a vehicle for innovation, or a means of fostering critical citizenship. Spirit operates as a conditioning influence, shaping the motivational frame (affect) and the interpretive view through which cognition processes mathematical concepts. This dual manifestation, visible in curricula (superstructural) and latent in values (substructural), exemplifies the hybrid interplay of hard and soft elements, as outlined in **Table 1**, where structured educational goals and adaptive cultural values recursively interact.

So, applying the metacybernetic framework to mathematics education, we gain a more detailed cybernetic realisation of how learning is structured across multiple levels. Recognising the interplay between cognition, affect, and spirit allows for educational strategies that are both structurally sound and ideologically coherent. This approach enhances mathematical proficiency and fosters a deeper connection to the subject, aligning educational practices with broader societal and cultural imperatives.

6. Case study: Application of substructure and superstructure in mathematics

Applying Mindset Agency Theory to mathematics education offers a framework to explore the interconnection between the underlying, conditioning implicate (substructure), potentially fostered by soft theories that prioritise inclusivity and deeper conceptual understanding, and the explicit, structured explicate (superstructure), often reflected in hard theories emphasising procedural fluency and efficiency. This case study examines how schools have balanced these dimensions in practice. Schools that effectively integrate substructural and superstructural elements in mathematics education tend to exhibit a dual focus: ensuring procedural competence while fostering adaptive, student-centred learning environments. A prime example can be seen in the Singapore Mathematics Curriculum, which balances highly structured problem-solving techniques (superstructure) with a strong emphasis on heuristics, metacognition, and inquiry-based learning (substructure) [37]. Using a structured visual approach to problem-solving supports conceptual understanding while promoting flexible thinking [38]. This scaffolding of mathematical structuring at both levels has been linked to higher problem-solving proficiency among students [37].

Similarly, Finland's education system embodies this balance by ensuring strong foundational numeracy (superstructure), while allowing for student autonomy and flexible learning pathways (substructure) [39]. The Finnish model moves beyond rigid procedural instruction by integrating collaborative problem-solving and real-world applications, fostering deeper engagement and conceptual mastery [40].

The impact of balancing superstructure and substructure in mathematics education is evident in measurable student outcomes. Studies indicate that schools, which incorporate rigorous problem-solving techniques (hard theory/superstructure) alongside adaptable, student-driven learning models (soft theory/substructure) experience higher levels of engagement, self-efficacy, and achievement [41].

For instance, research comparing traditional, teacher-centred mathematics instruction with constructivist, inquiry-based approaches in the United States found that students exposed to both structured procedural training (superstructure) and

flexible conceptual exploration (substructure) demonstrated greater retention, problem-solving skills, and motivation [42, 43]. This supports the argument that rigid proceduralism alone is insufficient—instead, the integration of ideational, reflective learning strategies enhances mathematical competence [44].

The case studies highlight that successful mathematics education systems align substructural and superstructural elements, ensuring operational rigour and adaptability. The application of MAT provides a metacybernetic lens through which these interactions can be modelled, offering insights into how schools can optimise learning environments. By mapping the interplay between hard and soft theories, educators can foster mathematical proficiency while cultivating deeper intellectual engagement, ultimately preparing students for academic and real-world problem-solving.

7. Implications and future directions

The conceptual distinction between superstructure and substructure articulated in MAT provides a useful framework to explore complexity in mathematics education. Superstructural elements correspond to the operational, technical, and procedural aspects of learning, whereas substructural elements encapsulate broader ideological, motivational, and cultural influences. This bifocal approach in education aligns with complexity theory, supporting the interconnectedness of cognitive, affective, and sociocultural dimensions [20].

For mathematics educators, the challenge is to integrate superstructural precision with substructural adaptability. Traditional mathematics education has often prioritised procedural fluency, equating success with mastery of algorithmic techniques [41]. While procedural competence is vital, research indicates that students benefit most when these foundational skills are complemented by heuristic reasoning, problem-solving strategies, and conceptual understanding [37]. Practical strategies for integrating superstructure and substructure include:

- Inquiry-based learning (IBL): Encouraging students to explore mathematical problems beyond rote computation fosters deeper conceptual engagement [38].

- Metacognitive scaffolding: Structured reflection on problem-solving approaches enhances self-regulation and cognitive flexibility [45].

- Growth mindset development: Dweck's [26] concept of growth mindset, situated within the broader Mindset Agency Theory [24], highlights the role of self-efficacy and motivation in sustaining learning over time.

- Technology-enhanced learning: Digital tools can support differentiated instruction, providing personalised learning experiences that cater to diverse cognitive and affective needs [46, 47].

By embedding such strategies, educators can create learning environments that support foundational competency and the ability to think critically, reason abstractly, and engage meaningfully with mathematics.

Policymakers play a crucial role in shaping the educational landscape by determining curricular priorities and assessment frameworks. Historically, mathematics education policy has emphasised standardised assessment metrics, often reinforcing

a narrow focus on procedural mastery [39]. However, a growing body of research advocates for a more balanced approach that accounts for cognitive outcomes and socio-emotional development [48]. Key policy recommendations include:

- Holistic assessment models: Moving beyond standardised testing to incorporate formative assessments, project-based evaluations, and portfolio reviews [49].
- Curricular reforms aligned with complexity Theory: Implementing policies that support dynamic, student-centred learning approaches while maintaining rigorous academic standards [20].
- Equitable resource allocation: Ensuring access to high-quality mathematics education for all students, including those from marginalised communities [50].

By encouraging policies that integrate superstructural and substructural elements, educational systems can enhance student engagement, resilience, and long-term mathematical competency.

The application of complexity theory and Mindset Agency Theory in education presents numerous avenues for future research. Key areas of investigation include:

1. Longitudinal studies on integrated mathematics education: Examining how superstructural and substructural balance influences student learning trajectories.
2. Comparative international analyses: Investigating how different educational systems implement these frameworks and their impact on student outcomes.
3. Neuroscientific approaches to mathematical cognition: Exploring how cognitive, affective, and spiritual dimensions interact at a neurological level [51].
4. Development of multidimensional assessment tools: Designing evaluative instruments that measure procedural competence and problem-solving flexibility, metacognitive awareness, and motivational factors [41].

By advancing research in these areas, scholars and practitioners can refine educational models that accommodate the complexity of human learning while ensuring meaningful engagement with mathematical concepts. Thus, integrating substructural and superstructural elements in mathematics education provides a comprehensive framework for fostering procedural mastery and conceptual depth. By drawing upon Mindset Agency Theory [24] and complexity theory [20], educators and policymakers can develop adaptive, student-centred learning environments that cultivate resilience, critical thinking, and intrinsic motivation. Future research might continue to explore the intersections between cognitive, affective, and sociocultural dimensions, ensuring that mathematics education remains both rigorous and responsive to the evolving needs of learners.

8. Discussion

Yolles' [24] framework emphasises the interplay between superstructure—the operational mechanisms that sustain a system—and substructure, which encompasses the ideational and value-driven elements that define the system's broader purpose.

This dynamic is particularly relevant in education, especially in mathematics, where structured curricula provide the foundational substructure, while the superstructure involves higher-order learning, metacognitive engagement, and the broader sociocultural purpose of education.

In mathematics education, superstructure manifests in rigorous procedural knowledge, algorithmic fluency, and the systematic development of core competencies (e.g. arithmetic, algebra, and calculus). Substructure, by contrast, extends beyond technical proficiency to incorporate reflective reasoning, inquiry-based learning, and ethical considerations in problem-solving. This dual-layered approach aligns with research on effective pedagogical strategies, where both procedural fluency and conceptual understanding must be developed simultaneously [52].

Further, the metacybernetic model suggests that systems achieving stability and adaptability must continuously adjust the balance between superstructure and substructure. In mathematics education, this translates to ensuring rigorous content mastery while fostering emotional resilience, collaboration, and curiosity—elements often overlooked in rigid, performance-driven curricula [41].

By framing mathematics education within the complexity continuum and substructure-superstructure principles, this chapter provides a strong theoretical and practical foundation for addressing the challenges educators face. It emphasises the necessity of balancing hard, measurable strategies with soft, value-driven approaches, ensuring a holistic, adaptive, and purpose-driven education system.

The distinction between hard and soft theories provides a useful means to recognise educational paradigms. These categories align with the substructure-superstructure interplay, shaping how mathematics is taught and learned:

- Hard theories correspond to structured, deterministic, and quantitative approaches. These models focus on explicit, measurable learning outcomes, standardised testing, and formal logic-based instruction. They reflect substructural characteristics, where stability, efficiency, and control mechanisms dominate educational design [53, 54].

- Soft theories, by contrast, introduce flexibility, recognising the importance of social, psychological, and affective dimensions in learning. These theories acknowledge students' agency, cultural context, and emotional engagement, aligning with superstructural considerations [41, 55].

The integration of hard and soft theories mirrors the superstructure-substructure interconnection, where rigorous content mastery must coexist with adaptable, student-centred learning. Yolles' work underscores the importance of balancing these dimensions dynamically, ensuring learning environments that foster both operational competence and epistemic openness.

One of Yolles' [24] core contributions is the recognition that cognition, affect, and spirit recur vertically across substructure and superstructure, ensuring coherence and adaptability in complex systems. This recursive integration is essential in educational contexts, where learning is not a linear process but an adaptive, multi-level phenomenon.

- Cognition operates at both levels: At the superstructural realm, it governs logical problem-solving and procedural fluency; at the substructural realm, it fosters metacognition, critical thinking, and creative problem-solving.

- Affect plays a pivotal role in motivation and resilience. At the substructural level, emotional responses to success, failure, and feedback loops influence engagement. At the superstructural level, deeper evaluative alignment with personal values and long-term goals fosters intrinsic motivation and self-efficacy [24, 26].

- Spirit, as conceptualised by Yolles [24], underpins the ideational foundation of a system. In mathematics education, this includes the values and cultural principles that shape learning. With respect to superstructure, it may be seen in curriculum design choices, while for the substructural realm, it manifests in the broader philosophical and societal goals of education.

Emphasis on recursive feedback loops and epistemic independence across levels suggests that effective mathematics education must integrate cognition, affect, and spirit in a self-regulating system. This aligns with complexity theory approaches in education, which emphasise adaptive learning environments, emergent knowledge structures, and reflexive pedagogical strategies [9]. Applying the superstructure-substructure framework and integrating hard and soft theories within a recursive model of cognition, affect, and spirit mathematics education can be reimagined as a dynamic, adaptive system. This approach provides educators with a multi-level strategy that balances rigorous content mastery, socio-emotional learning, and cultural relevance.

Future research should explore how these theoretical insights can be applied empirically, particularly in diverse educational settings where students' agency and epistemic diversity must be central considerations.

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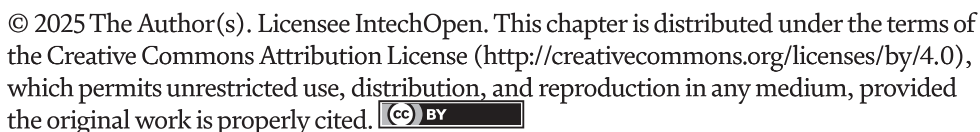

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Section 3

STEM Transitions: Challenges and Strategies for Early Science-Mathematics Integration

Chapter 5

Perspective Chapter: The Use of Mathematics in Chemistry and Difficulties That Arise for High School and College Freshman STEM Students

William Cary Kilner

Abstract

Chemistry is the gateway course for most STEM majors. Underprepared and at-risk students may struggle or fail, blocking further study in their majors and desired careers. They typically misunderstand the use of the formal mathematics they have studied in their previous mathematics classes as being what they will use in their study of chemistry. By disabusing them of this idea, we can pave the way for more success in this fundamental course. Students' formal mathematics training has been an abstract study of numbers and their relationships, whereas the mathematics used in the physical sciences deals with the manipulations of measured quantities of physical properties. Hence, we use the term, "chem-math" vs. "formal-math." Chem-math is the collection of mathematical skills from late arithmetic through algebra into precalculus that are needed for chemistry calculations. Students will not understand this differentiation by simply telling them. We must re-train them in basic mathematics manipulations they may have forgotten or never learned. These skills can be reviewed on an as-needed basis when chemistry exercises and subsequent authentic problems are introduced. We will then have students who lose their anxiety and better confront the material in their chemistry courses with a positive attitude, leading to increased learning and success.

Keywords: arithmetic, pre-algebra, algebra, advanced algebra, precalculus, problem-solving, exercises vs. problems, action-research, addressing mathematical difficulties, STEM careers

1. Introduction

This book chapter is excerpted in part from the thesis entitled "The Chem-Math Project," available on Proquest (Kilner 2014), and it represents the core of that work. Many other issues related to metacognition, the affective-domain, cooperative

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learning, constructivism, diagnostic tools for readiness, et al. can be found therein, and the 24 pages of references may be useful for instructors and researchers.

As a chemistry teacher in secondary and tertiary institutions for 33 years, I strove to document issues in mathematics that present great difficulty for underprepared STEM students. Chemistry is a course that many students fail, blocking further study in their desired majors. We hear students exclaim that "Chemistry is all math!" Life-science majors often show a great interest in science and avidly desire to work in their chosen fields. They may also think that the mathematics demands will be fewer if they aspire for a career in the life-sciences [1]. But students cannot avoid chemistry as it is a gateway course to all of the STEM disciplines [2, 3].

Many students in high school and matriculating to college may have struggled with the formal mathematics courses they studied in late middle school and high school. They think that the use of such mathematics, as an abstract and independent subject, is the important factor for success in the physical sciences. Mathematics facility is certainly important to all of the STEM disciplines. But students' formal mathematics training has been a separate abstract study of numbers and their relationships from the mathematics used to work in the physical sciences. Much mathematics research regarding basic mathematics skills can be found in the NCTM research publications (National Council of Teachers of Mathematics) and in their Yearbooks (**Tables 1** and **2**).

The mathematics used in chemistry and the other physical sciences deals with the measurement and manipulations of measured quantities of physical properties, hence the term "*chem-math*" vs. "*formal-math*." The relevant factor in chem-math is the ability to use basic arithmetic and algebra well. It does not require the higher mathematics needed for engineering, physics and upper-level chemistry courses. In high school college preparatory chemistry, and college introductory and general chemistry, many students use a myriad of incorrect and relatively simple procedures that they have either forgotten or were never well-learned in the first place. They move symbols around mechanically without understanding that the symbols stand for actual variables. This represents a lack of conceptual understanding of what is being accomplished in a calculation.

Consider another phenomenon seen from the affective domain [4, 5]. When students are desperate to pass a required course, they often revert to "Educational Darwinism" (as in evolutionary Darwinism). Unable to succeed by normal academic means, they panic and try all variety of irrational ways to survive in the course and true learning and understanding is lost [6]. One aspect of this stress-induced behavior comes from not understanding the chem-math tools they need to survive problem-solving requirements. By acquiring early instruction and practice in the necessary chem-math techniques, they will gain confidence in approaching problems to be solved [7–9].

NCTM reform references by date	
1989	Curriculum and Evaluation Standards for School Mathematics
1991	A Call for Change
1991	Professional Standards for Teaching Mathematics
1995	Assessment Standards for School Mathematics
2000	Principles and Standards for School Mathematics

Table 1.
List of NCTM resources that informed The Chem-Math Project and this present work.

1976	Measurement in School Mathematics
1978	Teaching Computational Skills
1979	Applications in School Mathematics
1980	Problem Solving in School Mathematics
1981	Teaching Statistics and Probability
1982	Mathematics for the Middle Grades (5–9)
1985	The Secondary School Mathematics Curriculum
1986	Estimation and Mental Calculation
1988	The Ideas of Algebra K-12
1991	Discrete Mathematics Across the Curriculum
1992	Calculators in Mathematics Education
1995	Connecting Mathematics Across the Curriculum
1998	Teaching and Learning Algorithms in School Mathematics
2002	Making Sense of Fractions, Ratios, and Proportions
2003	Learning and Teaching Measurement
2006	Thinking and Reasoning with Data and Chance
2008	Algebra and Algebraic Thinking in School Mathematics

Table 2.
NCTM yearbooks.

A "C" grade in a mathematics course represents less than optimum skill and understanding. This will be an impediment to progress in future mathematics and physical-science coursework. Previous weaknesses can be successfully addressed through extensive use of guided exercises while reteaching mathematical basics and providing a base of conceptual understanding. Such drill can be reinforced in small group-learning situations, where working closely with peers creates an esprit de corps [10, 11]. Students can also ferret out various mechanical and incorrect mathematics issues among themselves that the instructor might miss when dealing with the class as a whole, allowing remediation to proceed much faster. An example of such a formal program is POGIL, for *Process Oriented Guided Inquiry Learning* [12].

Students must realize that "exercises" are for practice, while "problem-solving" is a more complex skill that requires facility with their chem-math tools. Classroom and laboratory instruction should emphasize the conceptual basis for an exercise that is being executed. The instructor should demonstrate examples of how to "work it" (not "solve it"), and also identify possible confusing permutations that might be inherent to that exercise or application. Worksheets should be provided for student homework, explaining that learning these mathematical skills for use will be necessary to be able to solve word problems in the future. The credo of problem solving, as paraphrased by the grandfather of chemistry education research, George Bodner at Purdue, is that "problem-solving is what you do when you don't know what to do" [13, 14]. Therefore, it is incumbent that chemistry instructors at every level be cognizant of problematic issues in students' use of early mathematics, teach "chem-math" themselves as needed for their students, and not simply assume that students have the requisite facility from their previous mathematics instruction. Two useful computer

programs for individual student study are ALEKS (Assessment and LEarning in Knowledge Spaces), sold by McGraw-Hill (2025) and the Khan Academy (https://www.khanacademy.org).

Units (aka "dimensions," as in "dimensional analysis") are necessary to indicate what property is measured, in what measurement system, i.e. English or metric (SI), and the correct order of magnitude requiring scientific notation, i.e. the use of exponents. Instructors should begin by using simple word problems with basic formulae that define properties such as density, as well as equations such as the Ideal Gas-Law, that show the relationships among measured variables and constants. In this way students will come to see that "chem-math" is the application of basic manipulations found in early "formal-math."

Mathematics instructors, in turn, should periodically use numbers with units. This requires professional development activities between the mathematics department and the science department at any institution, so that each department knows and understands what the other department is teaching and what is needed for its students. This would involve the science department (teachers of chemistry) sitting side-by-side with the mathematics department (teachers of pre-algebra, algebra I and algebra II), and collaboratively comparing syllabi with each other's instruction [15]. Overall curriculum alignment can be addressed in a number of other ways using the *Understanding By Design* program [16]. There are other good books available on curriculum design and integration, particularly "curriculum mapping" [17].

2. History

This work began with 23 years of teaching high school chemistry, developing ideas from observation, then testing and using what worked. I presented these at local, state and national chemistry and physics conferences, gaining feedback from colleagues. Using the intervention method previously described I taught 5 years of weekly chemistry recitations for identified underprepared and at-risk students in college general chemistry. As I learned more about chem-math I continued to refine and target instruction, identifying further problematic examples of chem-math and undertaking additional study and analysis.

The research modality used is *action-research* [18, 19]. As qualitative research, the instructor is not separate from students but is a participant in a study. The instructor watches and listens intensely to students working, asking individuals what they are doing and why. Using photographed student work, informal and formal recorded interviews, and journal entries of observations of students at work, I have documented many examples of chem-math calculations that illustrate these issues. These are compiled as "The 27 Chem-Math Units," outlining the successful methodology that evolved to address weaknesses in students' use of fundamental mathematics.

When I first began teaching high school chemistry, I often heard from both students and colleagues that "Chemistry is *all math*" and that "Chemistry students struggle with the subject because they can't do *"the math"* [20]. My initial questions were, "What is this "math" that constitutes chemistry in the minds of teachers and students alike? What "math" are they unable to do?" Is this "math" considered the same by both students and teachers?" Much research in problem-solving focuses on proportional reasoning as the primary factor in student difficulties [21]. That, in turn, suggests larger issues regarding cognitive stages of readiness vis a vis conceptual understanding [22, 23]. However, other studies have shown that students with

adequate mathematical backgrounds can learn chemistry, regardless of their stage of cognitive (Piagetian) development [24–26].

An initial search revealed that a mathematics teacher had developed and taught a one-semester remedial mathematics course for 36 of 70 life-science students in a general chemistry course. She studied various high school and college chemistry and physics textbooks and listed the mathematics topics found in each and then used this information to develop her syllabus, which was published in her Ph. D. thesis in mathematics. She had also previously self-published a book on this work that I was able to examine [27, 28]. Although mathematics and chemistry instruction has progressed through several cycles over the years, we find her analysis of complementary subjects little different from the present.

When difficulties understanding mathematical operations and relationships arose, I took note of contextual specifics and methodically revisited mathematics fundamentals that were misunderstood or missing. Over time, I began to notice how I could structure necessary mathematical instruction in more meaningful ways. However, when I tried to discuss these discoveries with the mathematics faculty, I encountered little interest. They did not seem to appreciate how their mathematics was being used in science. One calculus/algebra II teacher responded "Oh, I could never pass your class! Chemistry is too hard!" When I asked another to help me with students' issues in learning chemistry problem-solving she said, "Chemistry is so hard; I could never do it!" However, a third teacher who focused more upon mathematical understanding rather than mechanical skill told me she enjoyed chemistry. Nevertheless, she and I were not able to create a working relationship since she was very focused on teaching and learning issues in her own subject. Often, I heard her students complain that she was "too hard" and was "making us learn proofs and derivations." Yet, when they returned from college to speak to her classes, they would thank her for much better mathematics preparation than their classmates had. Thus, I saw something useful in proofs and derivations that later became an important part of the Chem-Math Project, when I would have students learn to derive, rather than memorize, important equations they would use.

As I acquired a better understanding of instruction in formal mathematics from teachers with whom I consulted, I found it difficult for them to understand why it was important to better integrate our courses. As an example, I wondered why my students would not use the percent sign when multiplying a decimal fraction by one-hundred to obtain a percentage; they would simply tack the percent sign on at the end. When I asked the teachers about this convention, they told me "This is the way it is done," to which I explained that mathematical operations in the sciences need units to indicate what is measured, the nature of the mathematical operation, and to maintain an equality.

For these mathematics teachers it seemed the equality held. The percent sign was superfluous and it was merely added at the end as a convention, e.g.:

$0.34 \bullet 100 = 34\%$ was considered acceptable, whereas it should be written the following way:

$0.34 \cdot 100\%$, or $(0.34 \cdot 100)\% = 34\%$, since the percent sign is a symbol.

It is a label that indicates that we have moved from a decimal fraction (or a fraction of one whole) to a parts-per-hundred parts basis.

One mathematics teacher showed me that her students learned the following rule:

$0.34 \cdot (100/100) = 34\%$. But mathematically, when we divide 100/100, we obtain unity. And $(0.34 \cdot 1)$ does not equal 34%.

However, I later figured out what she actually meant by using this expression:

0.34 · (100/100) = (0.34 · 100 parts-per-hundred parts = 34 parts-per-hundred parts = 34 pph = 34%.

Since this apparent contradiction initially confused me, I felt it might confuse students further if I tried to explain that the expression they were taught was not truly an equality. The percent sign as a unit symbol must appear on both sides of the equals sign (unless it is factored out). Students need to understand this principle and not merely utilize dimensional analysis (aka unit analysis; the "factor-label method") mechanically in solving chemistry problems. It occurred to me that perhaps this inconsistency could lead to students' difficulty with percentages. They saw a mathematical expression with the percent sign apparently arbitrarily tacked on at the end and did not know what it represented. This is crucial because instruction in the use of unit analysis depends upon the careful selection and use of units as *actual algebraic variables* that must be conserved.

The percent sign is merely another way of stating parts per hundred (pph), which sets the stage for other small quantity ratios, ppm, ppb, and ppt, that also cause students trouble. As I continued to explore issues involving these conventions, I suspected students would have difficulties with the concept of a "decimal fraction," since many students were confused with mass-fraction, volume-fraction, mole-fraction and partial-pressure. Students did not understand that a fraction of one whole can be used in several different ways in chemistry. In my study of research in early mathematics instruction I learned that fractions represent a concept some students never truly master (NCTM 2002 Yearbook: *Making Sense of Fractions, Ratios, and Proportions*). Since unit analysis relies so heavily on fractions, I saw another place for a review when beginning to teach chemistry calculations.

A major premise of constructivism is that the acquisition of new knowledge is predicated upon prior knowledge. Thus, the teacher must first assess to find out what students *know* and are able to do in order to appropriately design instruction [29]. My emerging theory of chem-math is illustrated by the previous discussion, that unsatisfactory efforts to learn new material can result from earlier misunderstandings that were not addressed in previous instruction. Therefore, the instructor must also assess for what the student *does not know* if subsequent instruction is to be effective.

When confronting the phenomenon of the "MU" (a misunderstood *word, symbol, concept,* or *skill*) in lecture or in study on the part of the class or a student, it must be identified and addressed. Any teaching or reading beyond that point will be lost, due to the missing connection. In this example, a *symbol* i.e. the percent sign, often was not understood in previous instruction. Therefore, it would need to be addressed before these students would understand how to use percentages as well as ratios in chemistry calculations. A true cross-disciplinary integration of STEM courses is one of the original goals of the Chem-Math Project and a large component of national education efforts [30]. It is dependent upon gaining consensus on language and symbol use for important mathematical concepts used in the primary sciences of chemistry, physics and biology, as well as in the derived sciences; marine biology, earth science, astronomy and environmental science [31].

Although this vignette regarding percentage may seem trivial, it represents a core principle of the Chem-Math (CM) Project; that locating and addressing these seemingly insignificant issues can result in greater learning by removing blocks that prevent students from conceptual understanding, and that undermine their confidence in their ability to understand the subject.

As another example, my mathematics collaborator in high school disabused me of using certain mathematical phrases that had changed within the mathematics community since my own coursework as a student. When I told students to "Solve that exercise," some did not know what to do. I was instructed by my colleague to tell them, "Distribute terms, combine terms, and simplify." That cuing helped students to begin right away. When introducing mixture problems, I told students that you will have a situation with "simultaneous equations" or "two equations and two unknowns," and they would look at me blankly. My colleague told me to say "We will have a system of equations," and they went right to work. He also told me to use "rate of change" rather than "slope" when graphing. When students had trouble interpreting graphs on their calculators, I had them practice drawing graphs correctly on paper, with pencil so they could erase and not make a mess, and to lay their other written work out linearly down the paper. (Many students would randomly write all over the paper in a sloppy manner and thus totally lose track of a formal line of attack.) There is a certain precision to be learned as a part of formal-math that I had to learn to use myself from the mathematics community. Chemistry and physical science teachers using unit analysis habitually tell students to "Cancel those units." But my mathematics colleague told me to be more formal and rigorous. I was to tell them to "Factor out similar units," or "The fraction of identical units goes to one." In other words the units do not just magically disappear as in my percentage example.

3. Development

After extensive study, I could find no help for me or for my students in the many "mathematics for chemistry" skill-building books, or in the "math refreshers" typically located in chapter one or in the appendices of chemistry textbooks. These materials present the mathematics needed in a purely didactic fashion and do not provide the key insights students need to unlock and understand their difficulties. While these "reviews" and "refreshers" might appear to help, they merely "show" students what to do. They provide example exercises and problems worked out with the mechanics, but without attending to their requisite conceptual understanding. They explain the mathematics no differently from what is already found in mathematics textbooks – as purely mechanical operations. In addition, research in problem solving has shown that teaching via worked examples is not an effective strategy for students [22]. It merely fosters the mechanical approach antithetical to conceptual understanding. For instance, I once asked a student what he did when working on a difficult problem. He said he looked for an equation that contained the "letters" he needed, whereas he should be looking for an equation with the functionality he would need.

As I reviewed mathematical operations within my instruction, I explained to students what I was doing and why, and exhorted them to follow my examples exactly so that they would understand the importance of not skipping steps. But some students still struggled because no matter how patient and methodical I was in explaining, I was not reaching them. *Why was it that some students could understand and apply a mole-ratio the first time I presented it, whereas others found it confusing, even after repeated and varied presentations? Was the issue one of proportional reasoning? Was it the concept of the mole as representing a defined collection of unseen particles? Or was it the order of magnitude of this very large number?* I could find no answer in existing studies – only that it was "typically hard" for many students to understand the mole as the

lead in to instruction in stoichiometry (Latin: *to measure elements*). This key concept represents one of several quantitative stumbling blocks for neophytes [32]. I saw that students had a hard time rearranging equations. But when I derived the concept of the cross-multiplication of ratios (fractions) for them, and they practiced it in class, they were more able to handle equations without the tedium of multiplying both sides by the same term and factoring it out, or dividing both sides, etc. They also relearned they could invert both sides of an equation and the equality still held.

I also investigated physics education research (PER), since physics is more mathematics-intensive than chemistry. As I attended conferences and workshops, conversed with physics teachers, and read the literature, I found similar concerns about students' inadequate mathematics preparation [33, 34]. Arnold Arons is the grandfather of physics education, and much of this latter reference speaks loudly to the chemistry instructor. Various physics papers mention first-year students searching for an equation with given letters rather than understanding what equation is needed. Students in the calculus track would try to solve physics problems, but often the simple algebra gave them trouble. As with chemistry students they were using mathematics mechanically without conceptual understanding. This theme continues to be important in chemistry education research (CER) as well as in mathematics education research [35, 36]. The issue is not so much the mathematics itself, but how it is presented and used in novel contexts [37]. There is no inherent difficulty in typical chemical calculations. They are logical and straightforward and not tricky as are many mathematics puzzles. It seemed to me that it was possible to help chemistry students work chemistry exercises, then lead them into real problem-solving if given a proper scaffolding of skills and concepts. And as they obtained success, their confidence would improve.

Early in my teaching career, I assumed that students would understand calculations after first exposure. Disabused of that, I became meticulous in describing every step. As a result, I began to have more success teaching stoichiometry. However, I noticed that students were not as methodical as I was, despite my careful presentation and stern admonitions. They cut corners and skipped steps, something that I did not do as I wrote out the calculations that I always shared with them when preparing demonstrations. The issue appeared to be carelessness, an inability or unwillingness to follow instructions, or perhaps an unawareness of the importance of doing so. There also arose small and seemingly insignificant misunderstood words that I failed to recognize and clarify. This includes a number of chemistry terms that also had non-technical English meanings they would use incorrectly, such as "burn, boil, dissolve, mixture, substance," and "react to" when they meant "react with," et al. [31].

I realized that often teachers such as myself as a beginning teacher, gave up and moved on out of impatience, frustration, a lack of time and a "need to move on in the syllabus." Eventually I learned this to be counterproductive, since then students internalize bad habits while not obtaining the necessary foundation for later work. Therefore, I began to insist upon exact duplication of my instruction, and I invested the necessary time even if it held my lesson plans back. I developed a casual and humorous demeanor in order to be utterly insistent, while avoiding seeming dictatorial, and I attained greater success. As I scrutinized students' understanding of fundamental definitions and mathematics skills in greater detail and depth, and designed instruction around what I had found, my students gained additional success. Some students were obstinate, but if they were successful via an alternate route and demonstrated understanding, I let them use their own methodology, as long as the mathematics was correct. I sometimes asked them to share their work with the

class. For instance, I was surprised to find that students in my honors classes tended to prefer using a ratio-proportion approach for stoichiometry calculations – what we used to call the "equivalents" method [13, 38]. I had never taught it to them; they originated it themselves as a more logical approach. They saw the use of unit analysis as unnecessary, since it was merely another way of expressing the inherent proportional relationships they had already intuited. I would tell them that once they demonstrated to me that they understood unit analysis so that they would be able to assist classmates, they were free to use the ratio-proportion method for themselves.

I never heard much complaint about mathematics itself from my high school students because they were so well-taught by our mathematics department. I initially thought mathematics would not be such a large issue in teaching chemistry. Our mathematics department had codified and integrated their courses. All of the teachers were members of the National Council of Teachers of Mathematics (NCTM), and their syllabi were based upon the first edition of the standards [39]. My students could usually perform the formal mathematics they had been taught, regardless of whatever conceptual difficulties they might have had in chemistry. Nevertheless, I always had that cadre of "C" students in mathematics classes who struggled with the chemistry calculations that, for most of them, was the bane of the course.

I had extensive conversations with a mathematics teacher who had an engineering background. He taught the Algebra II course that many of my chemistry students were concurrently taking. One year I had a class of college preparatory students in Period 8 with their weekly double-period lab during Period 7/8. My mathematics colleague had the same students for Algebra II Period 6 and a double-period mathematics lab Period 6/7. Thus, during our mutual Period 7 we were able to visit the same body of students in each other's lab. He requested some word problems he and his colleagues could use to assist their students in chemistry and to provide a context for the mathematics they were learning. In turn, I suggested that the Algebra II teachers postpone teaching logarithms from November and move that instruction to April when we taught acids, bases, and pH. They acceded to this request and thus began a collaborative effort. Students understood pH more quickly in subsequent years and they were able to access more sophisticated instruction. The algebra teachers also moved quadratic functions closer to instruction in acid/base equilibria. This enabled us to provide a context for the use of the quadratic formula, rather than always defaulting to the 5% rule of approximation. I found it interesting that my students were happy to have an application for their formal mathematics skills and the chance to use them in a fresh context, providing an accessible challenge that they found intellectually satisfying. This became another piece of the Chem-Math Project—developing self-efficacy [4]. If students saw they were able to accomplish something that initially appeared to be very difficult, I could harness their emotional and intellectual enjoyment of such success to persevere in the face of future challenges.

Part of the emerging Chem-Math Project consisted of identifying the exact context for previously-learned mathematics skills and using the appropriate terminology. As the mathematics instructor and I sought other areas of overlap on which to capitalize, he enforced the use of the contemporary formal language of his mathematics instruction in our discussions. This was different from the terminology I was using, since it had changed greatly from what I had used as a student in high school. For instance, instead of using "slope," an abstract descriptive term, mathematics teachers now use "rate," providing physical meaning for this important concept. The NCTM standards recommend "rate-of-change" to prepare students for their study of calculus [40]. When I used the phrase, to "solve simultaneous

equations," or we have "two equations in two unknowns," students stared blankly at me. But when I said we would be using "a system of equations" as he told to do, they readily engaged in the calculation of isotopic abundances within mixture-problems. Students had recently been studying linear algebra using the matrix function on their TI-83 graphing calculators. They recognized its application to this chemical situation and spontaneously showed each other (and me) how to use the matrix program as a *numerical* way to solve mixture-problems, supplementing the *algebraic* and *graphical* solutions I had previously introduced. Then when students were able to describe this problem-solving situation *verbally*" (as they spoke with me and fellow students), we had used the four forms of mathematical representation recommended in the revised standards [40].

Mathematics teachers need to understand how mathematics is used in science, and science teachers need to know how students are taught their mathematics. I could see that issues of nomenclature were of paramount importance. If I were to access students' understanding of mathematics for application in chemistry, I needed to use the language in which they were being taught. Chemistry and physics teachers generally speak of "canceling" units when using unit analysis, and students internalize this phraseology. My mathematics colleague told me not to use this term, emphasizing that units do not magically "disappear." They "factor out," or "go to one." He felt that, from a student's perspective, if units can just "disappear" they must not be very important. Thus, careless language promotes careless thinking [41].

As another example, when I presented students with an algebraic equation and asked them to "solve it," I could not understand why they were unable to do so. Again, I was provided the algebra teachers' mantra: *Distribute terms, combine terms, and solve*. Once I used this terminology, I found that many more students were able to arrange equations correctly when problem-solving, because the phrase cued them into using the correct order of algebraic steps they had previously learned. Thus, I realized I needed to further investigate relevant mathematics education research. In the NCTM literature, particularly the Yearbooks, I found much useful material that subsequently informed my developing vision of the Chem-Math Project (**Table 2**).

When I began working with freshmen chemistry students at university, I discovered that preparation in a wide range of high schools is much more varied than I had assumed. Many students had far less facility in basic chemistry calculation skills, understanding of physical science fundamentals, and chemistry concepts, than my own high school students. I needed to assist students who had virtually no useful background to handle the depth and pace of college general chemistry. I also realized that the cadre of high-school chemistry teachers in local and regional groups in which I had been interacting were not necessarily representative of my college students' previous high school teachers. (Examples of such groups are the Seacoast Area Chemistry Teachers, the New Hampshire Science Teachers Association, the New England Science Teachers at MIT, the New England Association of Chemistry Teachers, and nationally the American Chemical Society Division of Chemistry Education.) By survey, many of the high school chemistry teachers of my college freshmen had a concentration in another science, with minimal to no in-depth chemistry background. As a result, they were unable to thoroughly prepare students in college-preparatory chemistry to ready them for their college general chemistry. My college students would sometimes tell me their teacher made their courses too easy so that all the students would pass. Or conversely, their teachers made the course too hard to scare everyone into working hard. I thus learned about *pedagogical content knowledge* or "PCK" [42] and the evolving Discipline-Based Education Research (DBER) movement to address such issues.

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Students ill-prepared from high school are at a great disadvantage when encountering general chemistry within a traditional format of lecture, laboratory, and homework (i.e. self-study). To address their difficulties, a college recitation program would need to have its own specific structure and not be a traditional problem-solving practice session run by teaching assistants demonstrating problem solutions on the board. Therefore, the Chem-Math Project recitation program evolved using mathematics themes derived from my high school instruction. Each category of chemistry exercise was isolated that required a specific mathematics skill. Then how the mathematics was to be used was described in each instance. I would write this up and discuss it with my mathematics colleague, calling it a "Chem-Math Unit." We began taking the first 17 of them to various local and national conferences. More issues were identified, leading to the current 27 Units. Thus, these came to represent a "typical" or "standard" high school college-preparatory chemistry curriculum.

Presenting early versions of the Units to colleagues for their scrutiny at conferences involved:

1. discussing a mathematical skill and illustrating the specific situations in chemistry in which that skill is used,

2. showing examples of how students mismanaged that skill,

3. presenting the pedagogical ways it has been successfully addressed, and finally

4. providing relevant research.

Follow-up discussion always ensued with great passion as colleagues contributed their own questions, problem-areas and ideas. Through a series of presentations and workshops, delivered both with and without my formal-math partner, feedback was obtained from other chemistry and physics teachers nationally and subsequently used to further refine these Units.

The Chem-Math Units
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4. "Chemistry is the study of the properties and composition of matter, and the changes it undergoes" (WCK)

This requires description, measurement and quantification. Students' performance in problem solving, along with their conceptual understanding and subsequent success in general chemistry, can be significantly enhanced by explicit instruction and practice in the specific mathematics skills they will use to work chemistry exercises and to solve problems.

With feedback from teachers I have developed the following 27 *Chem-Math Units.* Note the attention given to the affective domain and other peripheral issues, in conjunction with the acquisition and application of the requisite mathematics skills. As we go, notice where three separate locations of the concept of "scaling" exist and where the three levels of a deepening statistical understanding reside. Note that the

NCTM developed an alternative probability-and-statistics track to the traditional calculus track in higher mathematics that provides more access to additional mathematics study after algebra.

I make reference to the NCTM Yearbooks (National Conference of Teachers of Mathematics), which has been published yearly since 1926. Each year consists of contributed papers on that year's theme. I have found many of these to be very useful for understanding student issues in chem-math (**Table 2**).

5. The 27 currently identified chem-math units

5.1 Chem-math tools

1. Language, Semantics, Reading and Writing in Instruction
2. Using the Calculator Appropriately
3. Pattern Recognition

5.2 Chem-math basics

1. Rearranging Algebraic Expressions
2. Using Scientific Notation (Scaling #1)
3. Inter-converting Metric and English (SI) Measurement Units

5.3 Chem-math fundamentals

1. Unit analysis (aka Dimensional Analysis or the Factor-Label Method)
2. Unitary-rates and Derived Units
3. Ratio/Proportional Reasoning (Scaling #2)

5.4 Additional chem-math tools

1. Scientific Graphing
2. Significant Figures and Precision of Measurement (Chem-Statistics #1)
3. Percent-Composition, Percent-Error, and Decimal Fractions (Chem-Statistics #2)

5.5 Problem-solving

1. Authentic Problem-Solving vs. Exercises
2. Mixture-problems (Systems of Equations) and Weighted Averages (Decoding Word-Problems)
3. Estimation

5.6 Algebraic tools

1. Quadratic and Cubic Functions; Geometric Reasoning (Scaling #3)
2. Inverse-Square Relationships and Sign Conventions for Potential Energy
3. Exponential and Logarithmic Relationships
4. Rate-Problems
5. Complex Fractions

5.7 More chem-math problem-solving recitation materials

1. The Use of Proofs and Derivations
2. Formal Statistical Error-Analysis (Chem-Statistics #3)
3. Problem-Solving Heuristics, and the Correct Use of Algorithms
4. Developing Metacognitive Strategies
5. Reinforcing Study-Skills
6. Addressing Motivation
7. Using Models

6. Format for each slide

a. Why this skill is needed, or where it falls in the chemistry syllabus.

b. The pedagogical difficulty, or particular problem, this skill constitutes or presents for students in the chemistry curriculum.

c. Suggested handling and ways that have successfully addressed these problems in my own instruction.

d. Links to available research, and how it informs us about this particular pedagogical issue.

6.1 Language, semantics, reading and writing in instruction

a. Students must be able to decode the text and word-problems. They should be able to express their understanding of concepts clearly in their own words. They should consult a collegiate and/or technical dictionary at the first sign of confusion over a word or term. They should recognize common symbols for scientific terms and for mathematical operations. They should know that the Greek alphabet has upper and lower cases, and that these letters are different from English letters, even though some look alike.

b. Students' vocabularies are frequently poor. They do not understand the meaning and definitions of physical science nomenclature and of many key technical words used in chemistry. They do not realize the importance of precise definitions. They simply do not read.

c. Always define new words as they are used. Make students use these words in writing. Anticipate ambiguity. Use a collegiate and a technical dictionary during lectures, and read definitions rather than paraphrasing. If you generate a definition, formalize it and have students learn it verbatim. Never go past a word you suspect they might misunderstand without clarifying it.

Encourage students to use other texts to cross-reference material.

Students can now download a technical dictionary app on their phones. This requires they become more articulate in "addressing" their AI app.

d. Johnstone and Cassels' classic two studies in the early 1980s provide lists of common English words used in science whose technical meaning are different from their everyday usage. These are frequently misunderstood. Some examples are "burn, boil, dissolve, mixture, substance, react."

6.2 Using the calculator appropriately

a. Students only know how to use a few of the useful operations on their sophisticated graphing calculators. They are not facile with pencil and paper calculations and have become totally dependent upon calculators for arithmetic operations they should be able to do "in their heads."

b. Students do not utilize their owner's manual. They do not know how to use the memory function, or how the calculator stores previous operations. They do not utilize the very useful "2nd" or "inverse" button. They have used their calculators in their mathematics classes in a very different way from how they need to use them to practice exercises and solve chemistry problems. And they need to know how their particular calculator processes the order-of-operations.

c. Plan an initial training session with periodic practice sessions to address additional calculator skills as they are needed. Design exercises that take your students through commonly-used operations such as inverse functions and storing constants.

d. To the degree that students struggle with their calculators they will not have that energy available to solve chemistry problems.

See the NCTM Yearbook '92: *Calculators in Mathematics Education*.

6.3 Pattern recognition

a. This is an important link between mathematical problem solving and conceptual understanding; between algorithmic problem solving ("plug 'n chug") and conceptual problem solving (problem solving with understanding).

b. Pattern recognition is a vital part of chemistry (e.g. the periodic law, formula-writing, identifying reaction types). Some students have a native intuition for pattern recognition. Others can develop it via proper nurturing.

c. Show students how to discern visual, verbal, and mathematical patterns in various contexts. Provide them with exceptions to various patterns (anomalies) so they can see that exceptions do not necessarily negate emerging patterns. They will not then reject patterns out of hand when they encounter apparent contradictions.

d. There is a large body of research on pattern recognition in cognitive science and cybernetics, and some in mathematics education [43], but little in the problem-solving literature.

6.4 Rearranging algebraic expressions

a. This is a foundation chem-math pre-algebra/Algebra I skill.

b. Students cannot solve for a divisor or divide by a fraction. They forget their order of operations, the commutative rules, the distributive law and distributive properties. They do not realize that many common scientific symbols used in mathematics, science, and engineering come from the Greek alphabet, and are not the same as the English letters they resemble.

c. Administer an initial diagnostic quiz of basic algebra skills. Then do lots of drill and practice, remediating as necessary. Practice rearranging chemistry functions as math equations with letters and no numbers, using the proper symbols from the formulas and equations. Do not let them use "X's" for the unknown; there is already a letter symbol for it. Derive cross-multiplication with your students to demonstrate how it comes about from using correct algebraic manipulation. When students use the combined gas-law, have them group similar variables together (e.g. P_2/P_1), rather than randomly distributed within the function, in order to make the proportional relationship more apparent. Point out that symbols used for various quantities may be different in other texts and with other instructors, e.g. Q vs. Cp, q vs. ΔH, and the various upper and lower case "K's:" Keq vs. the rate-law "k" vs. the Beer's Law "k." Give students a handout of the Greek alphabet. Have them practice writing these symbols and help them find them on the keyboard.

d. Many students memorize their algebra without attaining a full conceptual understanding. They do not understand the meanings of "function, variable, equation," and "mathematical expression" [35, 44, 45] and the NCTM Yearbook '88: *The Ideas of Algebra K-12).*

6.5 Scientific notation (scaling #1)

a. Scientific notation is an essential tool of measurement. Scaling is a *Big-Idea* found in the National Science Education Standards [30].

b. Students do not understand "orders of magnitude;" they do not understand exponential scaling by tens. Students do not know the law of exponents, or

cannot transfer it from their abstract mathematics courses to science applications. They insist upon evaluating the scientific notation before operating. They do not understand "standard-form," and write the calculator "E-notation." They confuse this "E" with base e. They input "$10E^x$" and their answers are then off by a factor of 10.

c. Do not allow students to write E-notation; teach them how to translate E-notation into standard form. Instead of a rote rule for moving the decimal, tell students "when you put a number into standard-form, if you make the digits number smaller, use a larger power of ten," and "if you make the digits number larger, use a smaller power of ten." Thoroughly review the Law of Exponents, especially division with negative exponents. Have them watch the *Powers of Ten* video on YouTube.

d. Scaling is a component of proportional reasoning (National Research Council [30]. *National Science Education Standards*). It is a higher-order cognitive skill (HOCS).

Note that the NCTM standards (2001) do not include exponential notation.

6.6 Metric vs. English units

a. The use of the SI system and metric measure is a rite-of-passage into all of the STEM disciplines.

b. Our students have no intuitive feel for metric measure because they have grown up using English units.

c. Do not enforce "metric-only" rules; have students compare equivalent units. Do introductory measurement activities in both sets of units of mass, force, weight, temperature, and length, and calculate area and volume. Have students develop rule-of-thumb approximations for common conversion factors. Give them a table of conversion factors (both SI and non-SI) they will use in chemical calculations. Show the logic of a correctly-written measurement: the number provides the quantity, the prefix gives its power-of-ten magnitude, and the unit – the variable – states what property has been measured. By convention the metric number symbol uses lower case and the unit itself uses its first letter in upper case.

d. Most students have not manipulated, measured, and compared a sufficient number of physical quantities in their K-12 experience. They do not develop the intuitive feel for measurement that we expect for upper-level courses. Arons and Epstein have written extensively on activities to assist students in acquiring higher order cognitive skill (HOCS). See the *Introductory Physical Science* (IPS) text (3rd ed. 1977) and its physical science program for a well-tested model [46], as well as the NCTM Yearbook '03: *Learning and Teaching Measurement*.

6.7 Unit-analysis (aka dimensional analysis or the factor-label method)

a. UA is a very useful algorithm for scientific and engineering work, and is vital for understanding derived units. It can direct the computational process. Using multi-step UA minimizes the accumulation of rounding errors.

b. Many students learn to use UA mechanically without ever understanding *why* it works. Others never use it correctly; they do not understand *how* it works. Students often use slanted lines that gradually degenerate into vertical lines, so they lose track of the denominators and thereby miss a central purpose of the process. Mathematics teachers seldom use units and do not teach UA. They teach formal mathematics as its own subject.

c. Introduce UA early in your course. Use nonsense units as well as authentic exercises. Always require units with measurements. Teach that a "unit" is an algebraic variable, and is manipulated as such. I suspect that the "fence-pole" method of UA, often taught in secondary science courses, can promote a rote use of UA. Nevertheless, it enforces the rule to always use horizontal divisor-lines.

d. UA encourages many students to take a mechanical approach to problem solving. It can negate our best efforts at encouraging conceptual understanding unless we simultaneously reinforce proportional reasoning [36, 47]. Madeline Goodstein [48] has extensive UA instruction in her Sci-Math materials.

6.8 Unitary-rates and derived units

a. These functions define intensive properties, and also indicate the source of a variable.

b. Students do not know what "per" means. Their slanted lines degenerate into near vertical lines. (Point out that slanted divisor lines are acceptable in word-processing to eliminate justifying the following line of text.)

c. Provide a list of unitary-rates from physics and chemistry: velocity, acceleration, density, pressure, work, and power. Use UA to show how common SI units, such as the Newton, Joule, Watt, and Pascal are derived from definition. Repeat this derivation whenever students encounter a new unitary-rate. Convert "R" to energy units, and have students see that P·V is an energy term. Give them some mass and volume data for samples of a substance and have them graph mass vs. V rather than V vs. mass, as is usually done. This shows that the resulting slope represents the density (a unitary-rate) of that substance. Show that unitary rates can also be used as constants of proportionality, e.g. N_a, R, molar absorbance for a substance, the speed of light, Planck's constant, the Faraday constant, specific heat capacity, et al.

d. Unitary-rate is another facet of proportional reasoning. Many students never understand what a "rate" represents, despite the change in terminology from "slope" to "rate (of change)" in the NCTM mathematics standards. See the NCTM Yearbook '91: *Professional Standards for Teaching Mathematics*.

6.9 Ratio/proportional reasoning (scaling #2)

a. This skill represents formal reasoning in Piagetian terms. It is the basis for all of quantitative chemistry.

b. Students do not understand "lowest whole-number ratio." Students cannot distinguish between direct and inverse variation. Students cannot write a

mathematical function to represent a physical reality. In a classic study, students were given "There are 6 times as many students as teachers." Asked to write a formula for this, many students wrote it as "6·S = T."

Calculating with proportionality is a very different skill from calculating with UA.

c. Introduce the three gas-laws, presenting them with the two that use a quotient as a constant, and the one that uses a product as the constant. Then write them as the combined gas-law by combining constants, and show that the interaction of the three variables produces a new constant of proportionality called "R;" the universal gas-constant (PV/nT = R). Give lots of examples of proportions from physics as well as from chemistry. Make sure that students understand "specific gravity" and "specific heat" as ratios, and can see and understand that they have no units because they divide out (or "go to one") since the same units appear top and bottom. Ask which is bigger, a meter or a yard; a pound or a kilogram, a Celsius degree or a Fahrenheit degree beforehand. Then give them manipulatives so they determine this for themselves empirically.

d. Consult the NCTM standards concerning proportional reasoning in the K-12 math continuum, as well as the NCTM Yearbook '02: *Making Sense of Fraction, Ratios, and Proportions,* and the NCTM Yearbook '85: *The Secondary School Mathematics Curriculum.* There is a large body of research on proportional reasoning as being a higher-order cognitive skill. Students must construct this understanding, acquired through hands-on exploration. It requires content knowledge as well as pedagogical content knowledge from teachers throughout the K-16 pipeline [42, 49–53]. A particularly useful reference is *A Guide to Introductory Physics Teaching* [34].

e. Note: as per the NCTM recommendations, we no longer use *indirect* or direct *proportion,* but rather we use *variation.* And we do not use the term, *indirect,* but rather the term, *inverse.*.

Note: as per the NCTM recommendations, we no longer use *indirect* or direct *proportion,* but rather we use *variation.* And we do not use the term, *indirect,* but rather the term, *inverse.*

6.10 Scientific graphing

a. Understanding graphs is a cross-disciplinary skill, a vital scientific tool, and an essential component of proportional reasoning. It is one of the four ways the NCTM recommends representing mathematical concepts: graphical, symbolic, numeric, and verbal.

b. Students enjoy drawing graphs. However, they draw on autopilot, gleefully going through mechanical actions without understanding. They do not title them or spread them out, they do not label their axes, and they have trouble scaling them correctly. They use pens and make a mess because they cannot erase, and they feel compelled to place "x," "y," and "(0, 0)" unnecessarily on their graphs. They do not understand

that the slope of a straight line represents a unitary-rate. They cannot identify the dependent and independent variables even though they can define them. They do not know what residuals are when they use their TI-83^{+} to compute the curve-of-best-fit. They do not understand interpolation and extrapolation as distinct and useful skills for estimating data points not found on the line.

c. Provide students with a strict list of graphing rules to follow. Give them sample data to graph by hand. Have them interpolate and extrapolate to obtain data. Have them graph a nonlinear function by hand, such as VP vs. T, using a French curve. Show them how to transform a hyperbola into a linear function. Teach the basics of linear regression.

d. Note differences in expressing the slope-intercept formula in older textbooks as "y = mx + b" vs. "y = a + bx" in newer books, and ensure you use the form to which your students are familiar.

See why you must clearly differentiate the meaning of "b" in both instances. Students can graph ordered pairs, but fail to comprehend their physical meaning. Students confuse the y-intercept with the x-intercept. They do not understand the meaning of "slope." They cannot utilize values from a graph to form a correct interpretation of a physical representation [34, 43, 44, 54].

6.11 Significant figures (chem-statistics #1)

a. This is a vital concept and *Big-Idea* in the National Science Education Standards.

b. Students mindlessly report all digits from their calculators, and drop significant zeros after a decimal. They consider every measurement and calculation to be exact, because "that's the way it is in math class." They confuse a sloppy everyday use of "precision" and of "accuracy" with their technical meanings. Most students have had little contact with formal statistics, despite the recommendation of the NCTM Standards. Many students continually struggle to understand significant figures.

c. Have students refer to the rules of significant figures in the textbook whenever they compute, and understanding will come with repeated practice and exposure. Teach the rules for expressing measurements in one sitting and the rules for rounding calculations separately in another sitting, i.e. keep it simple. Tell them to "round to the least number of digits used in a problem to be solved," and it will work most of the time. Give them a worksheet with some ambiguous examples. This will show students why we need the rules and how they work. Explain that precision is a measure of the "fineness of measurement" whereas accuracy is the "closeness to an accepted value." In performing measurement activities, make sure they are aware of the limits of precision of their measuring instruments. Some good activities are calculating their individual horsepower, calculating the speed of sound, and calculating the period of a pendulum. (These are also good for exploring unitary-rates.) For advanced students, explore the propagation and accumulation of error in sequential calculations.

d. Students taught significant-figures didactically will not understand their use unless given frequent and sufficient experimental measurement situations in which to make rounding decisions. They will not understand the propagation of error until they see it for themselves. The *Journal of Chemical Education* has many references on the teaching of significant figures.

6.12 Percent-composition, percent-error, and decimal fractions (chem-statistics #2)

a. These concepts are important for overall numeracy, and for chemistry as the study of the composition of matter.

b. Students do not understand what "per-cent" means. They do not know what a "decimal-fraction" is. As a result they do not understand mass-fraction, mole-fraction, or volume-fraction, nor how these become percents. They do not understand that finding the "percent composition" means stating the percent of each constituent in a mixture or of an element in a compound. One does not "find the percent composition" of one constituent. Teachers can erroneously ask for the "percent composition of chlorine," rather that finding the "percent of chlorine in sodium chloride." Students use a calculator to multiply by 100 rather than just moving the decimal. They cannot calculate the whole when given the part and the percent.

c. Show students that "percent" means "parts-per-hundred;" that "%" is a unit meaning "pph." Remind them that "per" means a ratio. Percent is the number of parts out of the one hundred parts that constitute the whole. Have them use scientific notation in this manner: x% = (decimal fraction) · 102%.

   Define ppm, ppb, and ppt similarly, always expressing them in scientific notation. Teach percent-error as the accuracy of a result—the "closeness" to an accepted value. And use the *absolute value* to show it as the quantitative deviation from an expected experimental result, whereas the *signed value* tells which way it deviates. Define and distinguish between the validity of a measurement (comparing it to a "true" value) and its reliability (comparing it across multiple measurements). Ensure that students understand *calibration.* Have them calibrate thermometers in ice-water. Consult the NCTM standards and reinforce the teaching of statistics. If you introduce *standard deviation,* give them repeated opportunities to apply and practice it.

d. Mature students still struggle with ratios. Students do not know and are never instructed as to what the prefix, "per," means [34, 55]. See the NCTM Yearbooks 2006: *Thinking and Reasoning with Data and Chance*, and 1981: *Teaching Statistics and Probability.*

6.13 Authentic problem-solving vs. algorithms

a. This is the critical-thinking skill that represents formal reasoning as sought by all science educators. Exercises are for mechanical practice, whereas a "problem" is "what you do when you don't know what to do" (Grayson Wheatley via George Bodner). And "a problem that's not a problem isn't a problem" (H. Bent).

"Whenever there is a gap between where you are now and where you want to be, and you don't know how to find a way to cross that gap, you have a problem" [56].

b. Students learn algorithms but are unable to apply them to novel settings. They cannot see the stages in a multi-step word problem and bog down easily. They dive in without making a plan. They cannot filter out the "noise" in word problems. They do not understand "chunking" as information sequencing. They do not look for similarities to previous problems they have solved. They do not use an order-of-magnitude approximation as an answer check. They get frustrated. Unwilling to start over and try other stuff out, they give up easily. They do not review their solutions to see if they are "in the ball-park." They make a mess on paper and cannot follow their own work. They do not internalize what understanding they have acquired from successfully solving a hard problem [57].

c. Collect good problems from a multitude of texts and organize and group them to design a gradient of relevant transfer problems that unfolds with the syllabus. Students can gradually develop confidence, facility, and expertise through extended practice. An example would be to calculate what dilution to make of a concentrated acid to obtain a desired molarity, given the percent acid, density of the concentrated acid, and volume of product desired. Continually distinguish and connect the electronic, molecular, and molar perspectives [58]. Teach these traditional steps: "Understand the problem, make a plan, execute the plan, and look back to see what you did." [59]. Then make an order-of-magnitude estimation to see if the answer looks "reasonable."

d. There is much published research on "transfer-problems," the inability of students to move from exercises to problems, and the useless and haphazard approaches students invent when they do not know what to do [13, 57, 60–62]. Much discussion is found in inquiry-based instructional materials [15, 34]. It is also found in the knowledge-transfer literature [63]. See the NCTM Yearbook '80: *Problem Solving in School Mathematics.*

6.14 Mixture-problems (systems of equations) and weighted averages

a. The former are common word-problems found throughout the mathematics and science curriculum after Grade Six. They can be used to develop proportional reasoning. Some typical examples involve the price and composition of trail-mix, the density and composition of a two-component alloy, isotopic composition and atomic weights, and are found in dilution problems, calorimetry problems, and in "marathon" stoichiometry problems, such as: $C + O_2 \rightarrow CO + CO_2$ with mixed products, and $Cu_2O + CuO + C$ (heated to redness) $\rightarrow Cu + CO_2$ with these mixed reactants.

   Note that in these examples, students will want to go on auto-pilot and try to balance the equations as written. They must be disabused of this, since it is a different application of stoichiometry.

b. Students do not see the proportional relationships that are contained therein or the associated concept of weighted-averages. They trash the algebra and give up, or accept negative (nonsense) values without evaluation (see Unit 15).

c. Collect your own package of mixture problems. Develop skill by using them on a regular basis. Have students work in threes on portable white-boards and demonstrate various student solutions to the whole class. Have them work in pairs using the "think-aloud paired problem-solving" format (TAPPS). Record audio for later analyses (with student permission). Simultaneously use all four NCTM recommended representations: numerical, graphical, symbolic, and verbal. Remind students that they can use their calculator matrix function to solve systems of equations.

d. Mixture problems are used in middle school mathematics but are not effectively followed up in high school. See the NCTM Yearbook '82: *Mathematics for the Middle Grades (5–9).* These problems force students to use proportional-reasoning because there are two ratios embedded within one situation. Note: these situations are described as "systems of equations" in current mathematics texts. They are no longer called "simultaneous equations," or "two equations in two unknowns." Thus, students will not recognize them as such when so described [40].

6.15 Estimation

a. When solving difficult and multi-step problems it is important to make an order-of-magnitude estimation of an answer to ensure the result is at least in the proper "ballpark."

b. Students often produce ridiculous answers, far out of range or with a sign change, and accept them without inspection. Students think that *any* answer is better than no answer at all, whether it is meaningful or not. Teachers often give up on this issue out of frustration.

c. Teach students how to perform Fermi problems and give them a daily one for a while. This will provide them practice in estimating orders-of-magnitude. In addition, it will be a fun way to provoke thought and induce some critical thinking. Remind them of how to add and subtract exponents to estimate a result.

d. Estimation has always been a big problem that is frequently discussed in the problem-solving literature, given that students have been trained in their formal-math classes to obtain an exact result.

6.16 Quadratic and cubic functions; geometric reasoning (scaling #3)

a. This is another bridge between Algebra II and Chemistry & Physics. It constitutes an important understanding throughout the STEM disciplines and represents another aspect of proportional-reasoning.

b. Students do not understand the "pizza-problem;" they do not understand nonlinear growth. They do not remember the formulas for surface area and volume of spheres and of rectilinear solids. They see too few examples of these functions, since they are not grouped together in the syllabus and arise only occasionally. Students do not know where pi (π) comes from, and that it is a constant of proportionality.

c. Differentiate these problems from the linear functions with which students are more familiar. Use examples like the satellite dish/size of a telescope problem, and the heating of a large vs. a small building. Compare the area and volume of a cell vs. its diameter. Calculate the force crushing an empty alcohol can filled with condensing steam, and holding Magdeburg Spheres together after you evacuate the air from them. When you teach equilibrium, students actually welcome a chance to finally use the quadratic formula for something. So do not be in a rush to use the text for the simplification that produces a perfect square. However, this is a good place to teach the technique of iteration.

d. Lack of opportunities to practice these relationships across disciplines means that students cannot utilize their previous studies in geometry [34, 59, 64].

See the NCTM publication: *A Call for Change* (1991).

6.17 Inverse-square relationships and sign conventions for potential energy

a. Some examples of this function are Newton's law of universal gravitation, "candle problems" (light intensity), sound intensity, Coulomb's law, the period of the pendulum, and Graham's law of effusion.

b. While students may understand this function as an abstraction taught in Algebra II, they are not able to transfer it to these physical applications. Most texts give a very perfunctory treatment of Coulomb's law, a very important Big-Idea in chemistry. They also present it as an energy function rather than the force function, making it impossible for students to "experience" the function when manipulating the poles of magnets. In universal gravitation, Ep goes from a value of zero at reference to a positive number at infinite separation, whereas in atomic ionization energies, Ep goes from a minimum negative value at reference to zero at infinite separation.

c. Clarify these reference conventions for your students, since no one will likely have done so. Design a pictorial worksheet. Have students graph the above two functions, and in the correct quadrant, so they can compare them visually. Review the physics by deriving work from "W = f·d" and "f = m·a" to show from where the energy units come. Show how Ep becomes an inverse function, while force becomes an inverse-square function. Derive Graham's law, then have your students do the classic activity themselves for the experiment with ammonia and HCl vapor in opposite ends of a glass tube. Compare Coulomb's law and universal gravitation. But provide their relative magnitudes by reference to the four fundamental forces so that students understand the very different contexts in which these two functions are applicable. Make sure that you clarify the sign convention for attraction vs. repulsion. Also remind them there is no repulsion with gravitation as there is with electromagnetism.

d. The Chemical Bond Approach (CBA) high school chemistry text, *Chemical Systems* (1964), does an excellent job of presenting Coulomb's law early, referring back to it frequently in subsequent chapters. Such a discussion can usually be found in the small number of "principles" general chemistry texts. But it is poorly represented in standard college and high-school texts. There is some study of student difficulties with inverse functions in the mathematics education literature.

6.18 Exponential and logarithmic relationships

a. Some examples of logarithmic scales are the Richter scale, the decibel scale, the pH scale, the Beaufort wind velocity scale (in arbitrary scale units), and star magnitude (the base of this measurement = 2.5, not 10).

b. Students do not transfer understanding from their Algebra II course to these relationships, and they perform their calculations mechanically. They do not see the use of log-scales on a graph as a form of scaling. They want everything to be linear. Students in biology classes cannot draw the geological time-line in a long hallway with correct scaling. They incorrectly collapse (shorten) it without understanding the nature of exponential expansion.

c. Reteach logarithms as necessary. Show the *Powers of Ten* video several times. Ensure that students are comfortable using whole-number logarithms before using decimal logarithms. Review the law of exponents. Hand out log-tables and have students graph logarithm vs. its inverse on regular graph paper, then graph them again on semi-log paper and have students explain what is happening. Have them graph a table of natural logs (base-e) and explain why it looks similar to their initial base-10 graph. Ask them if graphing the logarithms to any base would look different. Show them how to use a proportional-parts table to obtain more precise log-values. Have them see how they could generate this table through interpolation of their graph of the log table. Give them log-log paper and an appropriate function, have them graph it, and then explain why it becomes linear. Show students a slide-rule and explain how it follows the law of logs. Have them graph $y = 10^x$ and $y = e^x$ and ask if they have the same shape. Note, this is a very long lesson whose work could be shared with an Algebra II/pre-calc colleague.

d. Science teachers do not generate opportunities to extend students' applications of logarithms into their disciplines. Conversely a lack of understanding of physical-science applications means that mathematics teachers cannot show when and how these functions are applied in the sciences. Students in precalculus often do not obtain a conceptual understanding of the exponential function.

6.19 Rate-problems

a. Some examples of exponential decay functions are radio-dating and drug-dose half-life problems, Newton's law of cooling, Beer's law using % T rather than absorbance, rinse problems, and first-order kinetics (including enzyme kinetics from biology).

b. These functions require more than a superficial understanding of the mathematics, so it is hard for students to use an algorithmic approach as a crutch. Mathematics teachers often fail to use these rich applications of the calculus in their own instruction. Students in calculus and pre-calculus will not see these connections in the material they are studying unless asked to derive these laws using experimental observations or tabulated data. Students do not recognize an asymptotic relationship; they often want the curve to hit the axis somewhere.

c. This is a great place to integrate mathematics and science. Kinetics is important in all of the science disciplines. Have students perform the classic "hundred-penny activity" (not the one for economics), and graph their results.

d. In the 1990's Texas Instruments and Ohio State University teamed up in the T^3 CasCalc Project to foster the use of technology to strengthen mathematics instruction. There is much research in calculus teaching and learning with these materials, especially the use of slope-fields [65]. But many mathematics instructors have their students use their graphing calculators in a mechanical fashion without an awareness of the rich body of pedagogy available that has been developed by this initiative.

6.20 Complex fractions

a. Some applications of complex fractions are the Clausius-Clapeyron equation, the van't Hoff equation and the related Arrhenius equation for activation energy, the Rydberg equation, calculating the reduced mass of a system, and finding the total resistance in parallel circuits and the total capacitance in series circuits.

b. Although students learned complex fractions in Algebra I, they have probably not used them in science. They will need this skill to calculate the energy of excited-state transitions in UV-visible spectroscopy, and in physics when they study electrical circuits and draw free-body diagrams.

c. With all that time intervening between initial exposure to complex fractions and your application, you will need to review this mathematics and perhaps give a pure mathematics worksheet on this skill.

d. See the NCTM Yearbook '88: *The Ideas of Algebra K-12.*

6.21 The use of proofs and derivations

a. Proofs and derivations have always been de rigueur in the traditional mathematics curriculum for developing enduring understandings. Using these in chemistry whenever possible moves students away from mechanical approaches and towards a deeper understanding.

b. When students do not understand what variables represent, or what an overall chemical equation or function represents, they try to memorize formulas. They "plug-'n-chug" on a survival basis despite our best efforts at teaching for understanding. It is hard to memorize a proof or derivation without understanding it. Repeated exposure to such proofs and derivations is necessary, since very little of this approach is found in contemporary mathematics and science coursework.

c. Look for opportunities to present derivations and proofs in your syllabus, particularly in the gas-laws and within the weak electrolytes unit. Present the combined gas-law, then "derive" the three individual laws as special cases by holding a variable constant. Derive the ideal gas-law from the combined gas-law and molar-volume. Solve it for "R," then use units to check the solution. Derive the gas-density formula from the density formula (d = MM/

Vm), and have students use it in place of PV = nRT when conditions are at STP. Derive Graham's Law from the formula for kinetic energy. Derive a formula for molecular weight from Urms (root mean-square velocity) or vice versa. Show Avogadro's proof of diatomic molecules, using HCl, H_2O, and NH_3. Derive K_a for acetic acid from K_{eq} for water and K for ionization; derive K_b for NH_3 from K_{eq} for water and K for ionization; derive K_h for NH_4Cl from K_b for NH_3, and K_h for $NaC_2H_3O_2$ from K_a for $HC_2H_3O_2$. Use pattern recognition in all these instances, since $K_a = K_b = 1.8 \cdot 10^{-5}$ (to two significant figures) for both this weak acid and weak base. Calculate the molarity of water in a dilute solution. Draw four "thermometer scales" and label them. Have students use them to visually derive the formulas for interconverting °C and °F, as well as for Kelvins and degrees Rankine. Then have students derive a new temperature scale based upon the freezing and boiling points of benzene (Problem #1.90 in the 5th ed., Silberberg, M., *Chemistry; The Molecular Nature of Matter and Change*).

d. There is still controversy regarding the role of proofs in mathematics instruction. This parallels the Saxon return-to-basics, and the phonics vs. whole-language arguments, over the issue of how teachers and curriculum developers can best find middle ground. This instructional model of using formal proofs does not seem to have reached chemistry and physics educators [66].

6.22 Formal statistical error analysis (chem-statistics #3)

a. Doing rigorous science requires an understanding of the reliability and validity of data. It is necessary to know the kinds and sources of error and their relative magnitudes, as well as ways to control it.

b. Middle school science teachers often neglect analyses of experiments, other than to acknowledge the presence of error. High school science teachers may have students calculate a "percent error" without inspecting for a possible source. Despite much instruction, students do not understand the control of variables (Arons). They associate simultaneity with causation. They think correlation means causation. They do not understand the propagation of error. They cannot conceptualize the difference between random and systematic error. Neglecting significant figures, they will express too little or too much error in their calculations.

c. If students have had some rudimentary instruction in statistics, they must be shown how to apply it to actual experiments. Every quantitative experiment should be accompanied by an attempt to ascribe sources of error and to formulate error bounds. At least one preliminary experiment dealing with measurement, some kind of calibration, and an estimation and calculation of error, should be done early in every science course so that students understand the process of science experimentation.

d. Students have had little exposure to useful statistics (although the NCTM is bringing Probability & Statistics into the mathematics curriculum as an alternative to the precalc/calculus track). *The Journal of College Science Teaching*, *The Physics Teacher*, *The Science Teacher*, and *The Journal of Chemical Education* all have articles regarding the analysis of error in experiments. The best references are

by Baird, Youden & Young. See the NCTM Yearbook '81: *Teaching Statistics and Probability.*

6.23 Problem-solving heuristics

a. Poor organizational and representational habits wreak havoc upon students striving to learn new skills in new settings.

b. Students tend to write too small and cannot see their work clearly. They lay out their work as if writing an essay, instead of giving themselves room to work stuff out. No one may have ever shown them the iterative process of problem solving. They use pen to write, cross stuff out, make a mess, and then cannot see their work clearly or follow what they have done. This visual disorganization contributes to cognitive disorganization. They do their work in the spaces and margins of anything handed out to them instead of starting their work anew on blank paper. They do not methodically rearrange any necessary formulas before plugging in numbers. They just start putting numbers in without units, and everything subsequently goes to hell.

c. Under constant scrutiny they will develop better work habits. But if not drilled regularly and sufficiently, they will revert to familiar bad habits when left to their own devices. Tell students to write LARGE! Provide pencils with a classroom pencil sharpener, and do not let them use pens; tell them to use pencil so they can erase. Hand out lots of scratch paper so they will use it! Make your worksheets so compact that they cannot begin to do work on them, and write on the top: "Please use fresh paper for your work." Walk around looking at student work, and "nag, nag, nag" until students consistently set up their work neatly and methodically. Remember that you are combating years of ingrained sloppiness.

d. Many problem-solving studies concentrate upon the next Chem-Math Unit (#24), and neglect these requisite mechanical aspects of problem-solving set-up.

6.24 Developing metacognitive strategies

a. Mature and successful students constantly observe, inspect, and evaluate their own thinking and learning processes.

b. Students launch into autopilot when given word problems to solve. They seek the "right answer" instead of the correct process. They try to find an equation that "fits," instead of following a logical progression of steps that leads them from current understanding to finding the appropriate relationship. They think an algorithmic approach is the most efficient route for problem-solving. They have developed this bad habit as a result of haphazard instruction in handling their mathematics in previous science courses. Your students may never have had explicit instruction in problem-solving.

c. Coach students into a careful reading of the problem, then assembling the information in an abbreviated format before plunging into calculations. Some teachers recommend "chunking" the info by reading five words at a time and determining what those five words are saying. Drill students in the following standard key

steps of the problem-solving process: Read carefully to understand the problem. Develop a clear plan to solve the problem. Do not neglect to consider recursion as an option. Execute the plan. Look back to see what you did. Check your work and identify what you learned that you can use in the future.

Have students practice the TAPPS (Thinking Aloud-Paired Problem Solving) procedure on occasion.

d. References by Polya, Schoenfeld, Simon, and Hayes provide an excellent research base on problem-solving. See the NCTM Yearbook '80: *Problem Solving in School Mathematics.*

6.25 Reinforcing study-skills

a. Good study-skills are essential for success in academics. They are a part of preparation for life-long learning, as well as for on-the-job professional development.

b. Every teacher should take some responsibility for assisting students in acquiring good study-habits. Precocious students may have never had to use them, but a sloppy approach may fail to serve them at some future juncture. Poor students may never have been properly coached. Students' proclivity to multitask means they may not have learned to marshal and apply their resources in a focused and concerted fashion. The university campus usually has a multitude of support structures in the sciences, mathematics, and formal writing of which students are often unaware.

c. Present a typical junior high school lesson in study-skills; most students still need it. Give them a list of outside resources for assistance in your course, including tutorial sessions, paid and unpaid tutors, where and when review sessions will be held, as well as good Internet resources.

d. The College Board has documented many ways in which students can improve their study habits. Use of an on-line intelligent tutor for mastery learning can reinforce the necessary focus on consistent practice so necessary for success in the STEM disciplines. One of the best is the ALEKS program www.aleks.com

6.26 Addressing motivation

a. Without a reason to learn, to master material, and to actually acquire true understanding, all instruction is for naught.

b. If a student's needs are not met, and/or if he feels that he is being dealt with unfairly, he will flip over into an irrational cope mechanism. He will seek to "get the grade" he can live with, and will focus all of his academic and intellectual attention on his other classes. I call this phenomenon *Educational Darwinism* (ED). Knowing he is unprepared, lost, and does not understand a thing, he panics and will do whatever irrational actions he thinks will keep him from failing.

c. Make a personal connection with each of your students. Discover their life goals. Let them know that you care about them, that you have their best interests in mind, and that your goal is helping them achieve their individual goals. Show

them how your course relates to their goals. Follow this basic rule of good teaching: "Find out what students already know and are able to do, and build upon that. Tell 'em what you're going to teach 'em, then teach 'em, then tell 'em what you've taught 'em." Make sure that your summative assessments test what you taught and not something else. Be flexible in dealing with student issues. The road to ED follows the law of least cognitive effort (Herron) in the attempt to survive and not fail a necessary course. You may periodically intervene by having students contemplate other possible careers they may wish to pursue. Frequently remind students that road to success in life follows focused effort and hard work in school.

d. Bandura [4], Dweck [67], and Pintrich [68] have written extensively about attitude and motivation.

6.27 Using models

a. Some of the best-developed STEM pedagogy is model-based instruction that builds upon the learning-cycle [51, 69].

b. Many students see science principles as a collection of unrelated statements to memorize. They do not see them as the result of attempts by scientists and scholars to make sense of and to rationalize observations of natural and experimental phenomena.

c. Consult the POGIL literature to see how lessons are preceded by a model of which students must make sense. The forerunner of this technique is the *learning-cycle* [51, 52]. Consult the physics education research literature to see how Hake and Halhoun used model-based pedagogy to revise the introductory physics curriculum.

d. The alphabet science courses of the post-Sputnik era, PSSC, BSSC, IPS, and Chem-Study, were all great exemplars of model-based pedagogy, especially in their laboratory programs where their models derive from experimental results. However, these programs require extensive teacher-training to ensure that they preserve the purity of the programs as they were developed and intended. In addition, PSSC and Chem-Study are not "programs for every student," since they are very difficult and do not serve the "average" student very well. (I cannot speak for BSSC, never having taught it. Note that it has a blue volume based upon evolution, and an alternative yellow volume based on molecular biology.) IPS requires pure dedication to inquiry-based learning. And since it usually falls in the eighth- or ninth-grade, this early course sets up a good basis for all of the upper-level STEM courses. With its hands-on nature of instruction, it is well-suited for students of all abilities.

7. Conclusion

As the more specific chem-math units were uncovered, more learning issues came to light. The role of affective factors in students' ability to focus, persist, and overcome difficulties is particularly important [70]. There are now 27 Chem-Math Units,

representing a multitude of issues related to success in learning chemistry, with mathematics still as the central focus. These could be used for professional development within the science department, and then shared with the mathematics department where the appropriate and most productive conversations and discussions would likely ensue. The Units could also function as a skeleton for individual chemistry/ physical science instructors to use in their own lesson planning.

The following format is what has been used in prior presentations.

1. A presentation of the entire list of the 27 Chem-Math Units can be shared via a Power-Point presentation 0f the Units, with or without discussion of their development, to stimulate interest and discussion. A paper handout of this whole package can provided for participants' consultation during the talk. If time is short, the participants can select a small number of specific Units of greatest interest.

2. Explain how the format section (A-D) organization will direct the audience discussion of each chosen Unit.

3. Show the Units chosen for discussion one at a time and open up the discussion. The presenter should function by answering questions and let the group do the talking. A participant could come forward to be moderator, helping to direct discussion since such discussions have proven to be quite energetic. Past audience feedback has resulted in the further development of the original 17 Units.

4. It has been impossible to get through all 27 of the Units at one sitting. Thus, such an intense presentation would best be scheduled for two separate sessions within a day, with a break in-between, such as a lunch, to allow for rest and informal conversations before resuming. This would enable the participants to think about the first segment of the presentation and come back to the second with more questions and suggestions. An all-day schedule would be to have two shorter AM sittings and two shorter PM sittings, given the typical intensity of discussion. For one sitting only, the presenter could show the whole list of titles and gather the list of the Units of most interest to the group, displaying for discussion only those that were chosen.

This presentation has been conducted at least twenty times and has proven to be a viable examination of problems that neophyte chemistry students typically have. Such a presentation would be appropriate for a high-school professional development workshop. It could be trimmed and revised for use in lower-level physical-science courses. For use in colleges and universities, it could constitute a course taken by graduate assistants who typically do not get serious training for providing meaningful recitation instruction. Or it could be used as a course for students in undergraduate STEM education programs [71].

Readers can certainly use these units as they see fit and adapt them to their needs. By using them in planning for instruction in exercises and problem solving, teachers may find relief from having to spend an inordinate amount of time explaining rather than teaching, when the concept and the skill are not brought forward in a coordinated and graduated way. The interested reader can find more information on the full ten-year chem-math study on Proquest under the title, "The Chem-Math Project" (Kilner 2014). Much more information can be found there on the integrated nature of

the Project to assist students in becoming more successful, including a full chapter on addressing students' conceptual difficulties with the course material in conjunction with the mathematics instruction.

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Section 3

STEM Transitions: Challenges and Strategies for Early Science-Mathematics Integration

Chapter 6

Exploring the Tech-Twist: Preservice Mathematics Teachers' Implementation of Educational Technologies for Visualisation at Resource-Constrained Schools

Mzwandile Wiseman Zulu

Abstract

The chapter explores the integration of educational technologies by preservice mathematics teachers to enhance learners' visualisation of mathematical concepts. It focuses on the transition from well-equipped teacher training environments to resource-constrained schools, highlighting the challenges and adaptations required for effective technology use in mathematics education. Grounded in the Technological Pedagogical Content Knowledge (TPACK) framework, the study investigates how digital tools such as GeoGebra, Sketchpad, and interactive visual aids contribute to mathematical understanding. The chapter also examines systemic barriers such as inadequate infrastructure, power outages, and limited access to technology, which impacts preservice teachers' ability to implement their training. The findings highlight the "tech-twist"—the adaptation strategies employed by teachers as they navigate these constraints while striving to implement digital pedagogy effectively. By analysing both opportunities and constraints, the chapter provides insights into the practical realities of integrating technology into mathematics classrooms, offering recommendations for bridging the gap between teacher training and real-world mathematics teaching contexts.

Keywords: technological pedagogical content knowledge (TPACK), preservice mathematics teachers, mathematics teaching, educational technology, visualisation of mathematics concepts

1. Introduction

The rapid development of digital technology in the modern era has opened up a growing range of new possibilities for incorporating technology into mathematics that may enable new approaches to both teaching and learning to improve learning outcomes [1, 2]. Numerous scholars have investigated the impact of these emerging

digital educational technologies on mathematics education, reporting both positive and negative findings, depending on the applications involved [3, 4].

One significant benefit associated with technologies designed for mathematics education is their ability to enhance the visualisation of mathematical concepts and problems [5]. Technologies, including manipulatives such as three-dimensional shapes, the abacus, and other counters, have been used since time immemorial to facilitate the learning of difficult concepts through active visual and tactile explorations. Digital technologies designed to facilitate learners' visualisation of challenging mathematical concepts may offer even greater benefits [6–8]. Learners' learning in mathematics is an active, iterative process that benefits from visual, interactive, and reflective strategies. By engaging with digital visualisation tools, learners construct mental models of abstract concepts, which reinforce their cognitive understanding and support long-term retention [7].

Several studies have found a positive correlation between the use of educational technologies in mathematics and improved visualisation of mathematical concepts by learners [5, 9]. For example, the interactive digital mathematics application GeoGebra has been found to improve learners' visualisation and mental schemata in the learning of geometry [10, 11]. Several researchers [12, 13] have demonstrated the efficacy of dynamic geometry software such as Sketchpad and GeoGebra to enhance the learning process in the context of transformation geometry, in particular. A South African study found that the use of GeoGebra enhanced learners' visualisation of concepts in Euclidean geometry, resulting in improved understanding [14]. Digital educational technologies that aid visualisation have also been found to benefit learners who experience language barriers in their understanding of theorems in geometry [15]. For fractions, digital educational technologies can be used to facilitate concrete visualisation to enable learners to gain a better understanding of concepts [16]. Some studies have found that learners often encounter difficulties calculating fractions without the aid of digital educational technologies such as calculators [17–19].

For the potential benefits of digital educational technologies to be fully realised, teachers must be equipped to use the range of technologies available and trained to integrate them into their teaching approaches. This places the emphasis on integrating the pedagogical use of educational technology into the training of student teachers. In addition, for teachers to utilise this training effectively, schools need to be equipped with the infrastructure and technologies that the teachers have been trained to use. Comprehensive training to enable preservice teachers to integrate various types of educational technologies from the beginning of their teaching careers is advocated by the United States Department of Education [20].

The inclusion of knowledge and skills for using digital technologies in teacher-training programmes is seen as crucial to meeting the educational demands of the current century. Several Asian studies [21–23] have highlighted the significance of equipping preservice mathematics teachers with the knowledge and skills to effectively integrate educational technologies into their teaching. A study conducted in Thailand found that preservice mathematics teachers who participated in a module on technical innovation demonstrated high levels of Technological Pedagogical Content Knowledge (TPACK) competence, indicating successful technology integration in their teaching [21]. In Vietnam, a study revealed that despite the availability of educational technologies at schools, the training that mathematics pre-teachers received was inadequate to enable them to use these technologies effectively in the classroom; this was due, primarily, to disparities in the ICT skills of their mathematics lecturers and limited access to advanced technologies at the higher education institution [22].

While many preservice teachers expressed eagerness to use digital educational technologies, some stressed the importance of continuing to use concrete, non-digital materials, particularly in resource-constrained rural school settings.

In Europe, the European Commission mandates that European teachers possess digital technology integration skills relevant to their subjects [24]. In line with this, teacher training institutions across Europe train preservice teachers in the integration of digital educational technologies with teaching and learning. However, a study involving nine European countries found that despite schools having access to various educational technologies, these were used primarily in passive learning, failing to harness their full potential to enhance teaching and learning [25]. In Spain, a study highlighted the role that lecturers can play in modelling the integration of technology to preservice teachers to encourage their implementation of digital educational technologies in the classroom [26].

In Africa, several studies have explored the use of digital educational technologies and the skills of preservice mathematics teachers to integrate these into teaching and learning [27, 28]. Some research has been done to investigate preservice teachers' self-efficacy in digital technology integration and the factors hindering their use of educational technologies in teaching [29, 30]. A study conducted in Tanzania showed that a professional development intervention led to improvements in preservice teachers' TPACK competence, particularly in the technology domain [27]. However, it raised concerns about why additional professional development programmes were offered during preservice teacher training instead of being integrated into core undergraduate coursework, suggesting that some teacher training institutions in developing countries may not adequately equip preservice teachers with essential digital technology integration skills. Nonetheless, continuous and consistent training for both preservice and in-service teachers is considered a valid approach to support the use of educational technologies in science and mathematics teaching in Africa [20]. In Botswana, Batane and Ngwako [29] found that while most preservice teachers reported high capability levels in using digital educational technology, they did not use these technologies effectively during teaching and learning, emphasising the need for a comprehensive and systematic strategy for technology implementation to help preservice teachers transition effectively into the school environment, where technology use is expected when available.

In South African studies, preservice mathematics teachers were found to lack pedagogical content knowledge for utilising educational technology, especially in the area of geometry [31, 32]. The reasons for these deficiencies were not explicitly stated, but a lack of institutional support during teacher training was identified as a contributing factor [33]. Jones et al. [34] investigated the impact of targeted training on preservice mathematics teachers to bridge the gap between conventional teaching and contemporary educational technologies. The study demonstrated a substantial improvement in participants' technological proficiency and confidence through reflective teaching practices, enabling them to seamlessly integrate technology into mathematics instruction. Similarly, Teane [35] addressed challenges faced by preservice mathematics teachers during technology integration training, noting barriers such as limited resources and varying technological literacy. Their scaffolded training approach significantly enhanced participants' adaptability and resourcefulness in utilising educational technologies across diverse classroom settings. Meanwhile, Olawale [36] work explored the impact of preservice mathematics teacher training on learner outcomes through technology integration, revealing a positive correlation between the depth of teachers' technological integration and learner achievement.

Emphasising the importance of both teacher competencies and the subsequent benefits to learners, the study highlighted the potential for enhanced learning experiences in mathematics classrooms.

The existing literature demonstrates that teacher training institutions focus on equipping preservice mathematics teachers with technology integration skills. However, there is a lack of research specifically investigating the use of educational technologies by preservice mathematics teachers for visualisation purposes, which is addressed in this study.

Since South Africa transitioned to democracy in 1994, the basic education curriculum has been revised several times in an attempt to address learners' educational needs and the educational deficits resulting from the legacy of apartheid. While the government has stated its intention that every learner should be proficient in information and communication technology (ICT) by 2013, insufficient funding for school infrastructure and technology hindered progress [37–39]. The high cost of ICT and the theft of expensive digital educational technologies have continued to limit the availability of modern educational technologies at government schools [1, 40, 41]. The lack of infrastructure and technology is particularly severe at schools in socioeconomically disadvantaged areas, hindering learners at these schools from developing technological competencies for life as well as accessing the benefits of digital educational technologies in the classroom [42, 43]. The COVID-19 pandemic highlighted the digital divide in South Africa between public schools and the well-resourced private schools that quickly adapted to remote learning; learners entering higher education from under-resourced public schools are thus even further affected by their lack of experience with digital technologies [44, 45].

The availability of digital educational technologies at South African schools and higher education institutions thus remains limited. At many schools, resource constraints have resulted in student teachers using educational technologies more for their own learning than for teaching, making it challenging to assess teachers' competence to integrate educational technologies into their pedagogy effectively [46].

Higher education institutions in South Africa also experience challenges with access to educational technologies. Ng'ambi et al. [47] identify distinct phases from 1996 to 2016, highlighting a growing digital divide along the fault line of the structural privilege and disadvantage that characterised apartheid. While the Department of Higher Education and Training (DHET) has introduced new policies promoting the use of ICT in education, the implementation of these initiatives has been hindered [47]. These policies, however, typically require preservice teachers to use their own electronic devices to access learning materials. Preservice teachers from different income brackets do not have the same access to devices, however, with the result that the use of ICT in education brought about a new form of inequity. The National Student Financial Aid Scheme (NSFAS) attempted to address this by distributing laptops to qualifying first-year preservice teachers; however, this provision was not implemented equitably across all public higher education institutions [48, 49].

Unequal access to digital educational technologies thus persists in the higher education sector, accompanied by a lack of knowledge and skills among instructors to effectively utilise digital technologies for teaching and learning [50, 51]. In this complex landscape, concerted efforts will be required to achieve the integration of educational technologies across all levels of education in South Africa and equip teachers to use these technologies effectively for teaching and learning.

Despite the challenges to the integration of digital educational technologies in the South African education landscape identified in the literature, teacher training

institutions remain committed to equipping preservice teachers with skills to integrate technologies into teaching and learning [52, 53], although substantial interventions are needed to ensure that digital technologies are available at schools to enable teachers to apply their skills [46]. Herein lies the technological twist, as preservice teachers who are prepared at the higher education level to integrate digital educational technologies into teaching and learning are confronted with a lack of appropriate technologies in the schools where they are placed for teaching practice.

Thus, based on the related literature, this study sought to respond to the following research questions:

i. How do preservice mathematics teachers integrate educational technologies into their teaching practices to enhance learners' understanding and visualisation of mathematical concepts?

ii. What challenges do preservice mathematics teachers face when transitioning from well-resourced training environments to resource-constrained schools, and how do these challenges impact their ability to effectively integrate technology into their lessons?

1.1 Mishra and Koehler's technological pedagogical content knowledge (TPACK) framework

Responding to the need for a grounding theory in the field of education technology, in 2006, Punya Mishra and Matthew Koehler put forward the Technological Pedagogical Content Knowledge (TPACK) framework [54]. Building on Lee Shulman's work on pedagogical and content knowledge, the TPACK framework adds the dimension of technological knowledge and introduces three intersecting knowledge domains: technological pedagogic knowledge (TPK), technological content knowledge (TCK), and pedagogical content knowledge (PCK) [54]. The intersections of these domains create areas of integration of content knowledge, pedagogical knowledge, and technological knowledge in the field of educational technology, as shown in **Figure 1**.

In the realm of educational technology and effective teaching, Mishra and Koehler [54] identify three key knowledge domains. Content knowledge (CK) encompasses the depth and organisation of a teacher's subject matter understanding, extending beyond facts and concepts to include the rationale behind specific concepts and their significance within a subject. Pedagogical knowledge (PK) is essential for effective teaching and includes skills such as classroom management, knowledge of learning theories, and the ability to motivate learners. Lastly, technological knowledge (TK) is crucial in the modern era, encompassing an understanding of traditional and modern technologies and the skills to use them effectively in education, given the influence of technology on teaching and learning. These three knowledge domains collectively form the foundation for effective teaching and educational technology integration [54–57].

In addition to the three key knowledge domains, Pedagogical Content Knowledge (PCK) combines PK with specific CK, addressing how to effectively teach a particular subject within the curriculum. The concept of Technological Content Knowledge (TCK) underscores the vital interplay between technology and educational content. It emphasises the importance of teachers comprehending how these two elements mutually shape each other so that they can make

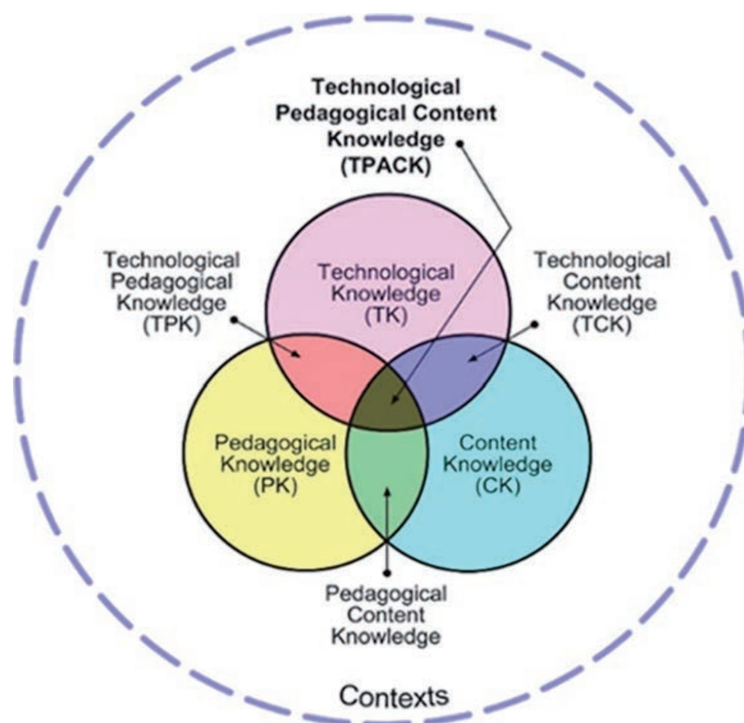

Figure 1.
Mishra and Koehler's TPACK framework (adapted from [54]).

informed decisions when selecting appropriate technologies to ensure the integration of diverse representations of content, ultimately facilitating effective subject teaching [58]. TPK works with technology pedagogically to transform the teaching and learning processes, building upon an understanding of TCK and the disciplinary context. TPK encompasses proficiency with various educational technologies and their impact on the educational process [57].

The integration of content knowledge, pedagogical knowledge, and technical knowledge in the TPACK framework transcends the individual components. It requires expertise in integrating technology effectively into the teaching and learning of content, applying pedagogical strategies, and facilitating learner learning. It encompasses a comprehensive grasp of technological skills, learner knowledge, subject matter content, and pedagogies to enable teachers to use technology effectively in the classroom, ultimately promoting successful teaching with technology [59, 60].

The TPACK framework is useful for preservice teachers and in-service teachers [61]. In this study, the framework grounds the exploration of preservice teachers' use of educational technologies for visualisation in their teaching.

2. Methodology

To investigate pre-service mathematics teachers' use of educational technologies as visualisation tools during teaching and learning, an interpretive, qualitative study was undertaken based on a subjectivist perspective, as data was derived from

participants' lived experiences and their connection to the real-world contexts they encountered. An adapted version of Schmidt et al.'s [61] TPACK questionnaire was used to gather data from 10 pre-service mathematics teachers who were purposively and conveniently selected as the participants for this study to determine their TPACK competence. These participants were then divided into two focus groups for discussions. The focused discussions were guided by 10 open-ended questions which assisted in understanding the participants' experiences.

Three weeks later, the participants were placed at schools for a month-long teaching practicum. The schools were located in 10 different communities across KwaZulu-Natal. Four of the schools were classified as Quintile 2, another four as Quintile 3, and two as Quintile 4[1]. School size ranged from 800 to 1310 learners, with teacher: learner ratios ranging from 35 to 65 learners per teacher. The mathematics average for the Grade 12 assessment ranged from 25.6 to 75.8%. No correlation was noted between Quintile, teacher: learner ratio, enrolment, and mathematics assessment.

Observations were used to gather evidence of participants' use of educational technologies during their teaching of mathematics in a classroom environment. As the timing coincided with the third wave of the COVID-19 pandemic in South Africa, the researcher was not permitted to visit the schools where the participants were placed; instead, their school mentors were briefed and then conducted the semi-structured observations [62] and video recording of lessons.

Table 1 provides a profile of the schools at which the participants were placed for their teaching practicum.

While validity and reliability are constructs associated with post-positivism to ensure the credibility of quantitative data collection instruments, this qualitative study employed credibility, dependability, confirmability, and transferability as criteria to ensure trustworthiness [63]. Additionally, the study incorporated member checking as a method to enhance trustworthiness [64].

3. Findings

3.1 Preservice mathematics teachers' TPACK

Participants' responses to the TPACK questionnaire demonstrated high levels of mastery for each knowledge domain, with mean scores ($\bar{x}$) of 4.2, 4.4, 4.5, and 4.3 (out of 5) for TK, TCK, TPK, and TPACK, respectively, as shown in **Table 2**.

These results indicate that almost all participants were found to be capable of integrating educational technologies into their teaching of mathematics. Some participants expressed confidence in their TPACK during the focus group discussion, correlating with the high $\bar{x}$ depicted in **Table 1**. One participant reported:

> *In my first year, I used an interactive whiteboard, transparencies, a projector, and a laptop. I also downloaded a video and played it in class just before I started a lesson to grab learners' attention.*

[1] In South Africa, schools are allocated to quintiles based on their resources and the characteristics of the community in which they are located, for the purpose of addressing disadvantages arising from South Africa's apartheid legacy through targeted approaches. The 20% of public schools with the least resources are assigned to Quintile 1, while the 20% with the most resources are assigned to Quintile 5.

School *	Quintile	Location	Enrolment	Teacher: learner ratio	Grade 12 mathematics assessment average (%)	Equipment
1	4	Pinetown	950	1:35	75.8	Whiteboard, Projector Textbook
2	2	Kwamaphumulo	800	1:35	28.4	Chalkboard Textbook
3	3	Umlazi	1100	1:50	64.7	Chalkboard Smartboard Tablet Textbook
4	3	Inanda	950	1:40	73.1	Chalkboard Smartboard Projector Tablet Textbook
5	2	Nquthu	1009	1:43	53.3	Chalkboard, Textbook
6	2	Nquthu	996	1:42	42.1	Chalkboard, Projector, Textbook
7	3	Bergville	1310	1:60	57.4	Chalkboard Textbook
8	2	Kranskop	1050	1:45	27.5	Chalkboard, Textbook
9	3	Ndengezi	1350	1:65	31.4	Chalkboard Textbook
10	4	Pinetown	861	1:38	25.8	Chalkboard Textbook

**School number corresponds to the number assigned to the participant placed there.*

Table 1.
Profile of schools.

Domain	Mean Score ($\bar{x}$) (maximum score: 5)
Technological Knowledge (TK)	4.2
Content Knowledge (CK)	4.2
Pedagogical Knowledge (PK)	3.8
Pedagogical Content Knowledge (PCK)	4.6
Technological Content Knowledge (TCK)	4.4
Technological Pedagogical Knowledge (TPK)	4.5
Technological Pedagogical Content Knowledge (TPACK)	4.3

Table 2.
Mean scores for each domain of knowledge.

Another explained:

> *I have used mathematics software this year... in a Grade 9 lesson, to teach them about transformation geometry. I was using Sketchpad from my laptop projected for learners... I showed the tracing of all the movements of the objects for them to visualise better on the screen.*

These results suggest that participants are proficient in integrating educational technologies into their mathematics teaching, reflecting strong TPACK. One participant's use of an interactive whiteboard, transparencies, a projector, and a laptop to enhance engagement highlights their TK and PK. Another participant's application of Sketchpad in teaching transformation geometry to Grade 9 learners showcases their TCK and the ability to merge technology with specific content areas to enhance learners' visualisation of mathematical concepts as also observed by Mukamba and Makamure [65] in their study.

3.2 Experiences with implementing tech in lessons during teaching practicum

When participants were placed at schools for their practicum, it was found that only two of the 10 schools had any kind of digital technology available for teaching and learning; two schools had smartboards; and one of these also had 30 tablets for learners and one laptop for the teacher. In terms of electrical equipment, three had digital projectors. One school had a whiteboard; the others used chalkboards. All schools used textbooks. Preservice mathematics teachers were encouraged by their lecturers to make use of any available educational technologies they found available at the schools where they were placed, as per their training.

The usage of technologies varied among participants due to availability. Some participants faced challenges which included the lack of supporting digital technologies such as projectors and speakers, but incorporated video clips into teaching using their personal laptops. Only two used specialised mathematic software such as GeoGebra and Sketchpad, which they brought with them on their laptops, in their teaching. Interactive charts were used by three, while one used the whiteboard only. Nine participants used chalkboards, highlighting the lack of more technologically advanced teaching technologies. One participant also expressed that they found these digital technologies easier and more enjoyable to use.

> *I enjoyed using it…the software [Sketchpad] made the teaching of the topic even easier… However, with the chalkboard, it was going to take time and maybe end up confusing.*

These findings illustrate that participants find digital technologies both easier and more enjoyable to use. The participant noted that using Sketchpad made teaching more efficient and clearer, contrasting it with the potential confusion and time consumption of using a chalkboard, demonstrating effective TPK. This highlights the participant's ability to leverage technology to enhance pedagogical effectiveness and improve the learning experience. Another participant, however, mentioned that the shift from the university environment to the school classroom can result in an unexpected 'twist' in their planned pedagogical approaches, as the infrastructure and technology may not be available for them to implement the integration of technology into their lessons in the ways they had planned:

> *At the university, we would just use technology anytime during peer-teaching. But when you go for teaching practice, things diverge when it comes to access to technology: we go back to the chalkboard.*

3.3 The usefulness and importance of non-digital tech to enhance visualisation

Participants were asked to discuss the potential for educational technologies to enhance learners' visualisation skills. They indicated that providing learners with

three-dimensional geometric objects to develop their visualisation skills could enhance teaching and learning thereby strengthening learners' understanding of mathematical concepts. One participant indicated that at lower levels of conceptual development, physical three-dimensional models that the learners can handle might be more effective than working with two-dimensional representations of three-dimensional shapes. Three participants stated that in a classroom situation where digital educational technologies were not available, other alternatives could be considered:

> *It depends on which grade you are teaching and what topic... For example, if you're going to teach 3D shapes in Grade 9, your approach should be different: bringing the actual 3D shapes in addition to the projected 3D images could be more effective for visualisation.*
>
> *Also, give them keywords of how they will then actually visualize these concepts when expected to...because the problem statements explain everything in words mostly, which you then need to visualize as you read...visualisation helps with concept development as well...*
>
> *Yes, it helps in many concepts...it could be algebra, geometry, or even number patterns: you can show them different patterns–to visualize the next pattern, and so on...*

These findings are supported by several empirical studies [66, 67] that have demonstrated that the use of geometric objects and manipulatives in the teaching of mathematics results in better comprehension for learners across different grades. Liggett [68] also corroborates that manipulatives serve as valuable tools, providing tangible means for learners to grasp abstract mathematical concepts. This underscores the importance of TCK in utilising physical and non-digital technologies to foster visualisation skills and conceptual development in mathematics.

3.4 Importance of assessing learners' prior knowledge

Participants emphasised the importance of assessing learners' prior knowledge before lesson planning to tailor the teaching approach to learners' needs. Yilmaz and Argun [69] established in their study that educational technology can be effectively integrated to visualise various mathematics topics, provided adequate preparation is undertaken before the lesson. This highlights the significance of TPK in combining technology with informed pedagogical strategies to meet learners' needs effectively, validating a conclusion by Schmidt et al. [61] when analysing the TPACK knowledge elements.

3.5 Benefits associated with using digital technologies for teaching mathematics

Observations of all 10 participants' lessons recorded that the various visual modes used by participants acted as a catalyst to promote an environment conducive to learning mathematics to varying degrees.

3.5.1 Validation of concept development in mathematics

All participants who utilised digital educational technologies noted that these technologies could be used to validate learners' conceptual development; this was

documented by their mentors during observation of their lessons, as well. One participant who did not have access to a projector in his school pointed out the following, recorded in the observation schedule:

> *Educational technology helps learners visualize the content being taught. For example, I could show learners that transversal cutting parallel lines form corresponding and alternate angles that are respectively equal to each other...this is something I had always wanted to do in Grade 10 ever since I was taught how to use Sketchpad… But the school didn't have a projector for me to do this better. I had to rely on my small laptop screen and make learners sit in groups to see; this strategy also worked.*

The participant's confidence in using digital technology in teaching is evident in this quote. This aligns with the idea proposed by Bonafini and Lee [70] that teacher educators should instil such confidence in preservice teachers as they enhance their TPACK skills. Participants' confidence in employing educational technologies for teaching geometry suggests a deep understanding of how these tools benefit learners during instruction.

One mentor documented the participant's successful integration of GeoGebra during a geometry class as follows:

> *His chosen educational technology influenced his teaching and concept development because he was able to convey the lesson expectations to the learners using the GeoGebra software that he brought with his computer. The use of his technology didn't get in the way of a successful lesson he taught which I'm sure he laid a good foundation for the understanding of the Grade 11 theorems he taught.*

According to the mentor, the preservice mathematics teacher exhibited strong TK by successfully using GeoGebra to convey lesson expectations and enhance concept development. Their seamless integration of technology into the lesson highlights their TPK, ensuring an enhanced learning experience for learners. The mentor's confidence in the preservice teacher's ability to lay a solid foundation for understanding Grade 11 theorems reflects the preservice teacher's good CK. This comprehensive understanding supports sustained conceptual retention for learners, as evidenced by Adelabu et al. [71].

3.5.2 Benefit of tech: Scaffolding learning

One of the mentors who collected data during observation of the participant's lesson documented the role that educational technologies played in strengthening the scaffolding of the lesson by engaging learners' attention and imagination:

> *Using visual resources during scaffolding in class: number 1, it draws learners' attention since they all want to feel like a part of the activity; and number 2, this isn't the same as narrating content to learners: the use of these well-prepared videos develops a good imagination for learners.*

Several scholars [72–74] have supported the incorporation of educational technologies as scaffolding in mathematics teaching based on their empirical research. Reinhold et al. [75] confirm that educational technologies act as scaffolds and thus can significantly aid learners' conceptual development.

3.5.3 Benefit of tech: Holding learners' attention

Other mentors observing participants' lessons also documented the impact of the use of educational technologies on sustaining learners' engagement throughout the lesson and extending their attention span. One recorded that:

> *[l]earners were focused on the lesson and happy to experience different teaching methods. I have never exposed them to a video lesson before. They concentrate better than they do if it is a chalkboard lesson…the 'normal way' we employ as older teachers.*

This underscores the importance of TPK in enhancing learner engagement by integrating diverse and effective teaching methods through technology.

3.6 Visualisation enablers for mathematics learners

One of the participants indicated that the use of digital technologies could enhance learning for learners with diverse learning styles; in particular, it could assist visual learners in developing mathematical concepts. Alabdulaziz [76] contends that the use of digital educational technologies in teaching and learning can facilitate the integration of visual, auditory, and kinaesthetic processes simultaneously.

When employed adeptly, GeoGebra has the capacity to enhance learning quality, enabling learners to visualise, investigate, explore, and build mathematical concepts [77]. The mentor who observed a lesson using GeoGebra made this comment:

> *I believe that the use of educational technologies by the student teacher influenced learners' abilities to visualize theorems because he was able to deliver concepts to learners visually with GeoGebra. His entire lesson using this tool was well executed.*

The mentor commended the student teacher for a well-executed lesson using GeoGebra, highlighting the effectiveness of TPACK in enhancing learners' understanding and learning experience in mathematics.

One mentor noted that in cases where digital technologies are unavailable, alternative options could still enhance learners' capacity for visualisation:

> *Our school is poorly resourced to assist student teachers who want to teach using modern technologies for things like the visualisation of mathematical content. With us old teachers, we just use our experience to ask learners to create mental images or put a sketch on paper when doing maths that requires some visualisation.*

This mentor demonstrated awareness of the need to nurture learners' visualisation skills and suggested a different approach to achieve this in a resource-constrained school setting where digital technologies are lacking. The mentor recognised the role of modern educational technologies in facilitating visualisation in mathematics learning, even though, as an experienced teacher, they were skilled in cultivating learners' visualisation abilities without relying on these technologies.

3.7 Implications of teaching in a resource-constrained environment

The data uncovered common challenges experienced by participants in the integration of educational technologies into their teaching. These findings suggest

a consistent pattern of difficulties across all 10 cases. For instance, the seven participants who designed lessons that included video material which they planned to project from their laptops found that the schools where they were placed did not have projectors that they could use to display the content to the whole class. Instead, they grouped learners around their laptop screens to watch the videos, although the volume was inadequate for learners to hear properly. One of the preservice teachers, who had a laptop and was placed at a Quintile 4 school for his practicum, had planned to use Sketchpad for his lesson but did not have access to a projector. He explained how he would have conducted the lesson with a projector and how he adapted his approach to accommodate the lack of access to this educational technology. Another participant's mentor reported the constraints she observed in this regard:

> *Her laptop screen was too small: the majority of learners couldn't see clearly. Since our school doesn't have projectors, she had to group them, and that was time-consuming. The lesson was delayed. That's why we just use chalkboards all the time.*

Thus, while some digital technologies (laptops) were available to support the integration of technology into the participant's pedagogical approach, the absence of other technologies (data projectors) to support this compromised the ability of the participant to incorporate technology effectively into their pedagogy in the lesson. The time required to set up and troubleshoot digital technology can also hinder teaching and learning, as the available time for a lesson is usually less than an hour. Consequently, some researchers [78, 79] have argued that integrating technologies into mathematics teaching could be counterproductive due to time constraints, particularly at schools with limited resources. In the example above, the mentor explained that experienced teachers continue to use chalkboards rather than transitioning to digital technologies as they lack the necessary resources to support the effective use of these technologies.

Six of the seven mentors who observed the preservice teachers using digital technologies noted the preservice teachers' efforts to find innovative solutions so that they could use digital technologies in their lessons despite the lack of resources at the schools were time-consuming, which had a detrimental impact on the teaching and learning process. Additionally, mentors' comments about reverting to chalkboards indicate that the kind of challenges they encounter impacts their abilities to effectively apply educational technology knowledge gained from training; they observed the preservice teachers encountering this, as well. For example, one participant's mentor reported:

> *…it was easy for chalk and chalkboard because learners are used to learning with those resources. The challenge started when the student teacher was using a laptop.*

The mentor reiterated the rationale for using a chalkboard instead of engaging with the challenges of adapting advanced technologies to resource-limited settings. This implies that the higher education curriculum might not fully acknowledge the resource constraints faced by basic education institutions in South Africa, despite higher education institutions having access to educational technologies for training future teachers. Consequently, preservice mathematics teachers often struggle to apply their university-acquired knowledge and skills practically in the school environment. These examples reveal the 'tech-twist' that teachers experience as a result of the disparity between their training in the use of digital educational technologies and the

constraints on its effective application in the classroom, highlighting the complexities of implementing technology in mathematics education within the South African context.

Macro challenges were also observed to impact the integration of digital technology into teaching and learning. Among these were the COVID-19 epidemic, which had a significant impact on teaching and learning at the time of this study, and the frequent power outages experienced regularly across South Africa. During COVID-19, the practice of sharing portable projectors was halted for health safety reasons due to the high transmissibility of the COVID-19 virus through skin contact. One of the mentors commented on this:

> *Before Covid-19 we used to have portable projectors that we would carry with us to class for some lessons and bring them back to our Head of Department offices. This system worked fine. But for health reasons, this system had to be halted.*

This had an adverse impact on the integration of technologies into the teaching process for resource-constrained schools.

Load-shedding, the practice of scheduled power outages, is an ongoing issue in South Africa that impacts schools, hospitals, and various sectors of the economy. Due to limited resources for implementing backup energy sources to support digital technologies, most schools are unable to depend on such technologies. Some of the participants experienced disruptions to their lessons due to load-shedding. One mentor reported the impact on the participant in terms of the integration of digital technology into his pedagogical approach:

> *He experienced a challenge since there was load-shedding during the lesson and that meant that he couldn't use the digital technology he started with*.

The disruption of this participant's lesson by a scheduled power outage resulted in a 'tech-twist', where the participant had to adapt their planned use of digital technology to accommodate unexpected resource constraints. This highlights the persistent challenges and limitations associated with integrating digital technologies in educational settings, as they are heavily dependent on a stable power supply, which is a critical issue in South Africa.

4. Discussion

The findings of this study align with the reviewed literature, emphasising how specifically educational technologies can potentially enhance the visualisation of mathematical concepts among learners. The preservice mathematics teachers demonstrated strong TPACK abilities, confidently integrating technology into their lessons, which was also consistent with previous studies indicating that comprehensive training in educational technology equips preservice teachers with the necessary skills [20, 21]. However, the study also focused on challenges confronted by preservice teachers applying the skills acquired from well-resourced training environments to resource-constrained schools, echoing Sahal and Ozdemir [22], who reported similar challenges in Vietnam. This suggests that while preservice teachers are prepared to use educational technologies, the lack of infrastructure in schools poses a significant barrier, resonating with Kafyulilo

et al. [27, 36]. Despite these challenges, the study revealed that preservice teachers effectively employed the limited available digital tools to enhance learners' understanding of mathematical concepts through visualisation, supporting findings by Bayaga et al. [10, 14] on the positive impact of tools like GeoGebra. In addition, this study contributes to understanding the dynamic interrelationship between technological innovation and pedagogical practices, further emphasising the necessity for infrastructure development in schools. Thus, offering the realisation of educational technologies in enhancing mathematics education.

5. Conclusion

This study offers a novel contribution to the existing South African literature by examining the integration of educational technologies by preservice mathematics teachers in schools where resource constraints pose significant challenges. By focusing on the transition from well-resourced training environments to resource-constrained school settings, this research emphasises the "technological twist" that preservice teachers experience and highlights the necessity of preparing teachers to adapt to varying levels of technological access.

The significance of this research lies in its empirical investigation of how preservice teachers leverage technological tools to enhance learners' understanding and visualisation of mathematical concepts, despite the limitations imposed by inadequate resources. The findings reveal that educational technologies, when available, can substantially improve learners' engagement and comprehension, thereby validating the theoretical underpinnings of the TPACK framework.

The data-based synthesis demonstrates that preservice teachers developed their TPACK competencies during their training but faced significant challenges in applying these skills in real-world classrooms due to resource limitations. This underscores the critical need for systemic interventions to bridge the gap between teacher training and school environments, such as strong partnerships between teacher training institutions and local schools, policy reforms, community engagement initiatives, and infrastructure improvements, all tailored to the unique needs of specific locations.

Finally, this study emphasises the necessity of adaptive teacher training programmes that prepare preservice teachers for the realities of diverse educational settings. These programmes would emphasise flexibility and resourcefulness, training teachers to effectively utilise limited resources, creatively implement low-cost teaching aids, and adapt modern pedagogical techniques to fit the realities of under-resourced classrooms. By addressing both the opportunities and constraints of technology integration in mathematics education, this research contributes valuable insights into the development of effective teaching practices that can enhance learning across various contexts. It further contributes to how the technological twist can be interrogated so that it does not contribute to maintaining and sustaining the present challenges confronted by preservice teachers in the South African context.

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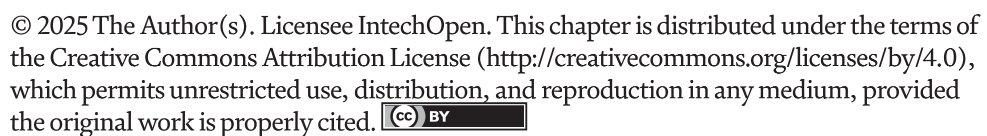
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Chapter 7

Perspective Chapter: Cloud Computing and the Laboratory of Numerical Mathematics

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Abstract

Universities and research organizations have cloud-based infrastructures, allowing users to access computing resources. This chapter describes how we can use cloud computing technology to create a numerical mathematics laboratory. With the help of cloud computing, we create a virtual computer suitable for performing complex mathematical calculations and creating auxiliary materials for mathematics education and research. We illustrate these properties of our technology with three simple examples. The virtual computer will be available in our university's cloud infrastructure. We perform the calculations using the SageMath notebook server, which uses the latest version of the Python language and runs on the Ubuntu operating system. The JupyterLab interface provides communication between the user and the server. Open-source systems ensure access to data and programs.

Keywords: cloud computing, virtual computer, numerical mathematics laboratory, sagemath, JupyterLab, Copernicus project

1. Introduction

We will briefly describe the content of our work. In the second section of our work, we describe the general challenges expected in university mathematics education. One such challenge is the adaptation to digital technologies. A recent literature review on this topic can be found in [1].

The third part contains the perceived challenges of mathematics as a scientific discipline. The work [2] shows the possibilities of the development of mathematical science using digital technologies. One such challenge of development is the use of open-source mathematical software. Such software is SageMath, see Ref. [3]. A detailed description of the program is available in the following publications, Refs. [4-6]. The software allows the creation of virtual numerical laboratories.

The fourth section includes the steps for creating such a laboratory. We are developing our virtual laboratory in the university cloud. In this cloud, the first step is to create a virtual computer. Here, we will install the necessary software. The installation consists of a few steps. These steps are instructions for Windows or Linux operating systems. A general knowledge of these systems is required to install the

virtual numerical laboratory. A general, theoretical description of the technologies used can be found in the work [7].

In the next section, we describe how we start working in our numerical laboratory. For simplicity, we will use a single-user system. Our laboratory uses the communication interface Jupyter and JupyterLab, see Ref. [8].

The following sections contain computer applications developed or archived in the numerical laboratory. The sixth section includes applications from integral calculus to illustrate mathematical education. Information about some other application areas can be found in the publication [9]. The system uses the Python programming language. Simpler codes can be run using a Sage Cell Server [10]. For more complex codes, such as integral calculus, it is possible to develop an HTML page for math education, see Ref. [11].

In the seventh part, we will briefly describe the Copernicus project [12, 13] and the possibility of archiving its applications. The Copernicus project is a BIG DATA technology that generates large amounts of satellite images. The generated images can be used to create applications, see Ref. [14]. This example shows how we apply our lab to another digital challenge for Big Data.

In the eighth part, we will examine how to use our numerical laboratory for mathematical research. How do we use artificial intelligence to prove mathematical statements using our lab? The part of our chapter deals with this question. The question is difficult, so we developed an HTML page in our lab [15] where we can perform various auxiliary calculations.

The last section summarizes the possibilities of using our virtual numerical laboratory.

2. Challenges in teaching mathematics

Teaching mathematics in technical higher education will face several challenges in the coming years. The most significant include:

2.1 Digitalization and technological changes

Digital tools and computer modeling are increasingly necessary in engineering and scientific study programs. The challenge is effectively integrating these tools into mathematics education while ensuring students continue developing strong theoretical foundations.

2.2 Diversity in students' prior knowledge

The level of high school mathematics education varies, meaning that students enter university with different levels of knowledge. Maintaining a uniform level in university courses is challenging and creates the need for preparatory classes to bridge the gaps.

2.3 Applied learning vs. theoretical depth

Technical students often focus on practical applications and may struggle to see the importance of theoretical mathematical concepts. The challenge is to find a balance between theoretical foundations and practical applications so that mathematics remains engaging and valuable for students.

2.4 Effectiveness of online and hybrid education

Since the pandemic, online and hybrid education have become common, but their effectiveness remains debatable. The challenge is keeping students engaged while ensuring a deep online understanding of the subject matter.

2.5 Development of mathematical thinking and problem-solving skills

Many students rely on rote memorization of formulas and procedures, struggling to develop abstract thinking and creativity in problem-solving. Mathematics education should encourage critical thinking and analytical skills instead of mere formula memorization without a more profound understanding.

An expanded database of these challenges can be found here [1]. These challenges require new pedagogical approaches and teaching strategies to help technical students effectively acquire and apply mathematical knowledge in their future careers.

3. Challenges for the science of math

Not only mathematics education but also the science of mathematics itself is facing new challenges, see Ref. [2]. Over the next 3–5 years, mathematics and mathematical sciences will encounter several emerging trends, challenges, and opportunities. Here are some key opportunities:

3.1 Artificial intelligence and mathematical research

Artificial intelligence (AI) is becoming increasingly integrated into mathematical discoveries. AI-powered systems are uncovering new optimization techniques and methods of computation. Automated theorem proving and computer-assisted mathematical research are also advancing.

3.2 Data science and statistical modeling

With the rise of data-driven decision-making, mathematical statistics, machine learning, and probability theory play an even more significant role in industry and science. New data analysis methods open new frontiers, such as modeling chaotic systems or a deeper understanding of nonlinear dynamics.

3.3 Interdisciplinary applications and mathematical modeling

We can connect mathematics with other disciplines, such as biology (genetic modeling), economics (financial mathematics innovations), and physics (new material models). Mathematical modeling enables more accurate predictions of epidemic spread and a better understanding of complex economic systems.

3.4 Open science and collaborative research

The spread of online collaboration and open-source mathematical software (e.g., SageMath, Julia) allows researchers to share their findings more quickly. Online proof verification systems and interactive research platforms help reduce errors and accelerate the validation of discoveries.

These opportunities will advance pure mathematics and elevate its applications to new levels in the coming years. In our work, we examine the application of one such challenge. One of the goals of our work is to show how we can use SageMath software to create a numerical laboratory. Such a laboratory will help us both in mathematical education and research.

4. A virtual computer

Nowadays, universities have high computing power, allowing cloud-based virtual computers to be created and used.

Our work describes the creation of such a virtual computer and some of its possible mathematical applications. We create a mathematical laboratory on our virtual computer, allowing various mathematical calculations. We can also use it to develop educational tools.

The following section describes accessing a cloud-based virtual computer within our university infrastructure. The description is a generalization of the procedure used at our university.

A system administrator manages virtual computers. The first step is to fill out an application where we define the fundamental properties of our computer.

4.1 Configuration of the virtual computer

Virtual computers are managed by a system administrator. As a first step, we need to fill out a request from where we define the basic properties of our computer. Parameters of our virtual computer:

Operating system: Windows 2025 Server
Number of processors: 4
Type of processors: Intel(R) Xeon(R) CPU E5-2670 v3 @ 2.30GHz
Memory: 8 GB
Disk: 200 GB

Another important thing is to set up the virtualization option on the machine, which is also the system administrator's responsibility. On non-virtual machines, we can set up the function using the BIOS. This function is needed because our math lab will run on Ubuntu Linux. Note that the operating system of our virtual computer could also have been Linux. We want to present a system that also uses Windows since it is a prevalent system that many people use. Our other comment is that in commercial cases, i.e., non-university systems, we can also access virtual computers using a similar procedure. After creating the virtual computer, the administrator sends the user the login information. We will use the virtual computer within the university infrastructure.

4.2 Configuring the mathematical laboratory

One of the goals of our work is to show how we can use the SageMath software to set up a mathematics laboratory. Therefore, this chapter will describe how to install the system on our virtual computer. We assume the dear reader has basic IT knowledge of the following: Therefore, we do not go into detail about the individual steps of our procedure. This guide contains information for the following system

software versions: Windows Server 2025, Ubuntu 24.04, SageMath 10.5, and Python 3.11. The procedure for creating the mathematical laboratory consists of two main steps. In the first step, we create a virtual Linux operating system on our computer, such as Ubuntu. It is necessary because our mathematical software will run under this system. In the second step, we install the SageMath system.

4.3 Installing Ubuntu Linux using WSL

Windows operating systems (from version 10) allow you to create virtual machines under the operating system using the Windows Subsystem for Linux (WSL) function (assuming that virtualization is enabled on your computer). We suppose that you are logged into your virtual machine. We will bold the commands that we will perform. We will number the installation steps. Instructions for creating an Ubuntu Linux virtual machine:

Powershell: launching Powershell and followed by some Powershell commands
wsl –list –online: list of possible operating systems
wsl –install –Ubuntu-24.04: install Ubuntu 24.04
wsl -d Ubuntu-24.04: start the OS

After the system's first boot, you must enter the Ubuntu system administrator and their password. Then, we are updating the system with the update and upgrade commands. Our administrator's name is the *dministrator* for our computer. Our administrator's name is *dministrator* for our computer *WS205-Tuke*. We will need this when entering the commands later.

4) Create an account for dministrator (administrator for Ubuntu)
Unix USER name:
password:
5) dministrator@WS25-Tuke:/mnt/c/Users/administrator$ *sudo apt update*
6) dministrator@WS25-Tuke:/mnt/c/Users/administrator$ *sudo apt upgrade*

4.4 Installation of SageMath in Ubuntu

The installation of our SageMath system can follow. However, before that, we need to install some utilities.

7) dministrator@WS25-Tuke:/mnt/c/Users/administrator$
curl -L -O "https://github.com/conda-forge/miniforge/releases/latest/download/Miniforge3-$(uname)-$(uname -m).sh"
8) dministrator@WS25-Tuke:/mnt/c/Users/administrator$
bash Miniforge3-$(uname)-$(uname -m).sh
9) dministrator@WS25-Tuke:/mnt/c/Users/administrator$
eval "$(/home/dministrator/miniforge3/bin/conda shell.bash hook)"
We will install the latest version of SageMath using the Conda system. Here we need to specify the version of the Python language, which will also be installed.
10) (base)dministrator@WS25-Tuke:/mnt/c/Users/administrator$
conda create -n sage sage python = 3.11

These were the steps in the creation of the virtual mathematical laboratory.

4.5 Launch of the numerical math laboratory

Steps to start the laboratory:

11) Log in to the virtual computer (Windows) in the university's cloud.
Note: If the reader wants to log in from home, they must first log in to the university's network using a virtual private network. Because our virtual laboratory is located in the university network.
12) Booting the Linux system.
13) (base)dministrator@WS25-Tuke:/mnt/c/Users/administrator$
eval "$(/home/dministrator/miniforge3/bin/conda shell.bash hook)"
14) (base)dministrator@WS25-Tuke:/mnt/c/Users/administrator$
conda activate sage
15) (sage)dministrator@WS25-Tuke:/mnt/c/Users/administrator$
sage -n jupyterlab

These commands will start the JupyterLab server. This procedure (steps 12–15), like the system's installation, can be automated. Presumably, such automation, with appropriate parameterization, will make things easier for users.

This section shows that creating a private virtual mathematical laboratory based on open-source mathematical software is not that easy. We have described every part of the process in more detail.

5. Working in the virtual laboratory

Based on the previous section, we start our laboratory. We start the JupyterLab server so we can perform interactive calculations. This interface begins automatically when we install an Internet browser (e.g., Firefox) on our Unix system. If not, we can start with an Internet browser in our Windows system and enter the http://127.0.0.1:8888/lab command in the command line.

After starting our virtual laboratory, we can begin work. If we want to implement a job or project, it is advisable to name the project and create a program library. **Figure 1** shows that within the system, we have created a project called *Integral Calculus, Copernicus project*, or *AI and Goldbach's conjecture*. These projects serve as examples of the application of the numerical laboratory. Of course, many other applications are possible.

We can create program codes within the directories to perform calculations, evaluate satellite images, or illustrate a mathematical object or theorem. We develop programs using Python or SageMath. We wrote the code for this chapter in *SageMath*. The system, therefore, displays outputs interactively and online. That means we can specify and modify the input parameters for our calculations and present the calculations on the Internet.

With minor modifications, we use some programs from the book [4], *An Introduction to SAGE Programming with Applications to Sage Interacts for Numerical Methods*. A detailed description of the mathematical capabilities and applications of SageMath can be found here [5]. The online application technology of interactive calculations is included in the publication [6], *Sage for Undergraduates: Second Edition*. The authors have previously encountered the open-source SageMath software [3]. The article [9] describes their experiences with its use. Nowadays, new technologies such as cloud computing make it possible to create a virtual laboratory for SageMath.

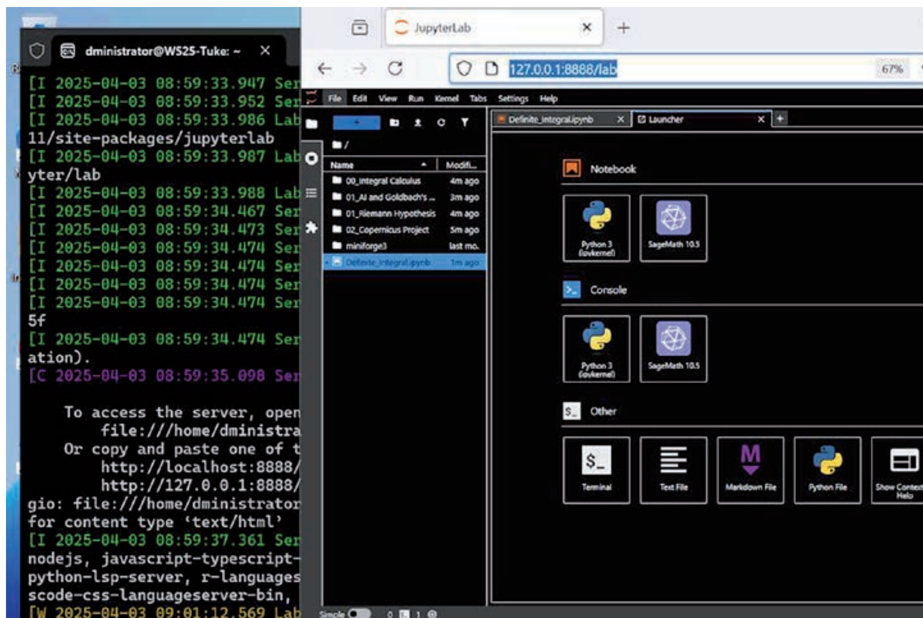

Figure 1.
Organizing mathematical projects after starting the numerical laboratory. Creating the project or job directories.

We can perform calculations that do not require multiple partial online; see Ref. [10]. In this case, setting up a virtual laboratory is unnecessary.

6. An application for mathematics teaching: Integral calculus

It is necessary to adapt to digital technologies. Integrating artificial intelligence, online tools, and interactive platforms into mathematics education requires updating teaching methods and adapting to evolving technological trends. In our virtual laboratory, we can create presentations and teaching materials that demonstrate mathematical knowledge. A simple example of such teaching material is integral calculus. Integral calculus is part of mathematics education at all technical universities.

Our project works with three basic concepts of integral calculus. These concepts are the Riemann integral, the fundamental theorem of calculus (Newton-Leibniz Formula)—the definite integral calculation, and the numerical calculation using the trapezoidal method. The topic naturally includes several other important concepts (such as improper integrals), so we can expand the project as desired.

We can develop programs to visualize mathematical concepts and algorithms. These programs run directly within our system, and **Figure 2** displays an example output. The figure plots a function $f(x)$ over the interval $[a,b]$. We divide the interval into n segments and draw the corresponding rectangles, illustrating the Riemann sum for this partition. When running the program, users can freely choose the input values:

- The function $f(x)$
- The interval $[a,b]$
- The number of divisions n

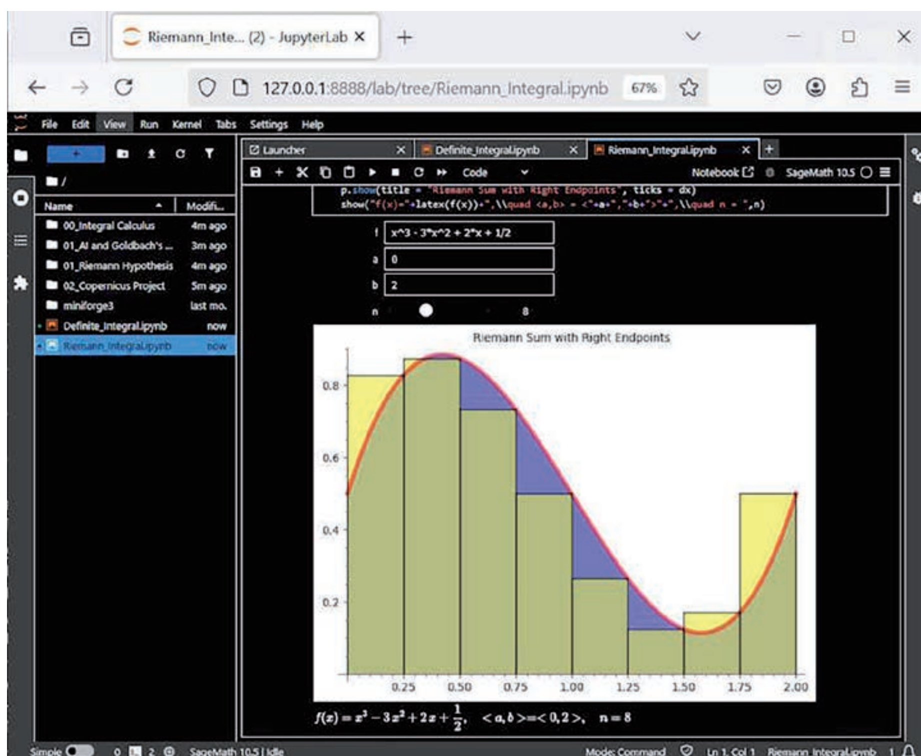

Figure 2.
Creation and presentation of teaching materials. How does the Riemann integral works?

6.1 Presentation of projects on the internet

We want to showcase our project on the Internet and run our SageMath or Python programs there. Since our SageMath and Python code is open-access and includes interactive features, users can provide input data for the programs. To make this work, we can embed our code in an HTML file, which allows the program to run directly in a web browser. A detailed guide on this process is available in Ref. [6]. We should place the source code in a modifiable cell when embedding programs in HTML. This way, if newer SageMath versions introduce changes, we can easily update the code. While such compatibility issues are rare, we should still account for them. Users can also adjust program parameters when running the code, allowing them to perform calculations with valid input data.

6.2 Source codes of the integral calculus

The presentation of our integral calculus project can be found on the Internet, see Ref. [11]. If we call up the HTML page, three cells of our project appear, in which the source code of the Riemann integral, Newton-Leibniz formula, and the trapezoidal rule for numerical integration can be found. The programs can be started with the Evaluate button below the cell.

These source codes were tested in our virtual laboratory. After tuning the programs, the programs were transferred to an HTML file and stored on the Internet.

7. The Copernicus project: Big data archiving

This section describes how we can work with Big Data technology in our laboratory. Big Data technology presents another challenge for mathematics education and science. The Copernicus project produces satellite images and a vast amount of data. The system has over 300,000 users and works with $91x10^{15}$ bytes of data. Project users can create programs for their projects and save data using the JupyterLab interface. However, due to the huge storage requirements, these data and programs cannot be stored continuously within the Copernicus system. Therefore, system users can store and archive their projects, programs, and data in their numerical laboratories. A brief description of the Copernicus project can be found in this work [14]. A satellite image of one of our projects, where we observed highway construction, can be seen in **Figure 3**.

We briefly describe the Copernicus project as a representative of Big Data technology.

7.1 The Copernicus project

"Europe's eyes on Earth"—this is the motto that welcomes you when you visit the Copernicus website [12]. It captures the essence of the program's mission: to provide a comprehensive view of our planet from space. The Copernicus project is an Earth observation program managed by the European Commission. It monitors the entire planet, with a specific focus on European territory to benefit Europe's citizens. The program offers open access to nearly 91 petabytes of data—a figure that continues to grow. The project relies on the Sentinel satellite constellation, which consists of

Figure 3.
Archiving programs and other outputs from external projects. A Copernicus project satellite image of the construction of a motorway near Košice, Slovakia.

Satellite	Focus area	Example applications
Sentinel-1	Radar imaging (C-band Synthetic Aperture Radar)	Surface change detection, floods, and ground movement
Sentinel-2	Multispectral optical imaging (10–60 m resolution)	Vegetation health, agriculture, and urban monitoring
Sentinel-3	Ocean and land monitoring (temperature, color)	Forest fires, water quality, and sea surface temperature
Sentinel-4	Atmospheric monitoring (geostationary, future)	Air quality, gas emissions
Sentinel-5	Atmospheric monitoring (onboard *MetOp* satellites)	Ozone, NO_2, CH_4, and other trace gases
Sentinel-5P	Precursor with TROPOMI sensor	Global air pollution monitoring
Sentinel-6	Sea level measurements (altimetry)	Climate change, sea level rise

Table 1.
Overview of Sentinel satellites, their focus areas, and example applications.

six satellites, each equipped with specialized monitoring capabilities. Each satellite observes different parts of the Earth (**Table 1**).

Data from these satellites are processed into six thematic fields of observation: Atmosphere, Marine, Land, Climate Change, Security, and Emergency. However, this source of information is not limited to these fields. The large volume of raw data can be processed in various ways, enabling a wide range of applications and development opportunities. The primary data sources used in our analysis are images provided by Sentinel-1 and Sentinel-2. A comparison of these satellites is presented in **Table 2**.

7.2 Access to data

To access Copernicus satellite data, the first step is to register on the official platform [13], which is available for free. After completing the initial login, users must visit the Copernicus Data Hub dashboard to create an OAuth client. These clients

Feature/parameter	Sentinel-1	Sentinel-2
Type of sensor	Synthetic Aperture Radar (SAR)	MultiSpectral Instrument (MSI)
Imaging type	Radar (active sensor)	Optical (passive sensor)
Operation in weather conditions	Works in all weather, penetrates rain, haze, and clouds	Sensitive to clouds and atmospheric disturbances
Day/night Capability	Day and night	Only day
Resolution	10–20 meters	10 meters (for selected bands)
Revisit time	3–5 days	Max 5 days
Monitoring focus	Oceans, forests, surface movement, soil, and infrastructure	Land surfaces, vegetation, soil, and water body changes
Possible application in aviation	Detects terrain, infrastructure height, and works through fog	Observes land use, vegetation near airports

Table 2.
Comparison of Sentinel-1 and Sentinel-2 features and parameters relevant to Earth observation and aviation.

enable secure login and authorize the downloading of satellite imagery and other relevant data from the system.

For more advanced data processing tasks, JupyterLab is a highly efficient and versatile interface provided by the platform. It supports a range of file formats, including notebooks, text documents, code scripts, and Markdown files. JupyterLab runs on virtual servers and supports programming in Python, allowing users to develop, test, and run code directly in the cloud environment.

The platform offers three types of virtual servers, tailored to different levels of computational demand. These servers are capable of handling large-scale data processing tasks. Users are provided with 10 GB of cloud storage space through the MyStorage system, where they can save their notebooks, datasets, and code. MyStorage also supports folder creation, file sharing, and uploading of locally stored content and applications.

Because the total number of registered users is 340,855, the system automatically deletes stored data if the platform is not accessed for fifteen days. Although logging in during this period is sufficient to maintain access, it is highly recommended to also store important files and applications locally to avoid unintended data loss.

7.3 Downloading files from the platform

Individual files and images can be downloaded easily by right-clicking on them and selecting the download option. Unfortunately, the system does not support downloading entire folders directly through the user interface.

To overcome this limitation, we can use a simple Bash command within a Jupyter Notebook to compress folders into a single archive file (e.g., in .tar.gz format), which can be downloaded as a single file. This method is both efficient and straightforward.

8. Artificial intelligence and Goldbach's conjecture: Some research computation

We can also use our virtual laboratory for mathematical research. One interesting challenge is communication and interaction with artificial intelligence. We have dealt with defining mathematical problems for artificial intelligence so that the technology could try to prove the given mathematical theorem. We determined that it is necessary to transform the mathematical theorem into a form that the technology can interpret.

We want to try a new way of proving some number theory problems with this technology. We have prepared some mathematical statements for the technology and some conjectures that need to be proven. We have described our conjectures using programs. Here, we state our conjecture that every natural number $n > 3$ can be written as the sum of two or three prime numbers. Our program contains such an algorithm, which calculates the decomposition of any natural number into the sum of two or three prime numbers, see Ref. [15]. The problem is related to the Goldbach conjecture.

The example illustrates how our virtual mathematics laboratory could serve as an application for solving problems in number theory with artificial intelligence.

9. Conclusions

Universities and research institutions operate cloud infrastructures that provide users with access to computational resources. This chapter describes how cloud

computing can be used to create a virtual laboratory for numerical mathematics. The goal is to establish an environment suitable for complex mathematical computations, the development of educational materials, and research support.

Our implementation leverages the university's cloud infrastructure, where a virtual machine equipped with the SageMath computational server is deployed. The system is based on the latest version of Python and runs on the Windows server WSL Ubuntu operating system. User interaction is facilitated through JupyterLab, which provides an intuitive interface for communication between the user and the server. All technologies used are open-source, ensuring transparent access to data and software.

The installation process for the numerical laboratory has been described in precise, step-by-step detail, allowing users to follow the procedure without requiring advanced IT knowledge.

The practical applicability of our platform is demonstrated through three concrete examples, showcasing its potential in both education and research.

Our virtual laboratory serves as a versatile tool for mathematics education and the creation of supplementary learning materials. For instance, in our integral calculus project, we demonstrate key concepts such as:

The Riemann integral,

The fundamental theorem of calculus (the Newton-Leibniz theorem), and.

Numerical integration techniques.

This project can be further expanded to include advanced topics like improper integrals and indefinite integrals, enriching its educational value.

Another critical application of the laboratory is data and software archiving, particularly for large-scale projects like Copernicus. Given the project's extensive user base (thousands of researchers), efficient storage management and controlled data access are essential. Our system addresses these needs by implementing structured storage limitations and access protocols.

The laboratory also supports mathematical research, including emerging fields such as AI-assisted theorem proving. An intriguing open question is: How can artificial intelligence be trained to prove mathematical theorems? While we only touch upon this challenge in Section 8, it presents a compelling direction for future work.

Our virtual laboratory is well-equipped to address a wide range of research challenges. Notable examples include supporting the development of publications such as *Computational Mathematics with SageMath* [5] (available in open access format) and *Computer Approaches to Mathematical Problems* [16], which features numerous computational problems that can be effectively solved using a virtual laboratory environment.

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Chapter 8

Perspective Chapter: Learning Mathematics through Graph Theory Games

Merrill W.Y. Yen, Kenneth Y.T. Lim and Richard Y.J. Lee

Abstract

Graph theory is the study of graphs which is a mathematical structure that helps to model relationships between objects. Graph theory has applications in diverse fields such as computer science, biology, sociology and transportation. Thus, graph theory allows us to understand relationships across different fields. This makes it imperative to call for educational materials to engage students to learn graph theory. Since graph theory is highly visual and has low barriers to entry making it a good field to explore game based learning, we aim to explore the design and development of games to help to keep students engaged and interested with graph theory concepts. Participants will engage in a series of games with underlying graph theory concepts infused in it. An interview to examine the effects of how the games has brought about understanding in the graph theory concepts will be given after the workshop. The games are expected to increase the participants' engagement with graph theory by making the learning process more interactive and enjoyable, improving their understanding of the concepts introduced and the real world applications of graph theory. The intended impact of this project is that students come out with a comprehensive knowledge of graph theory and an understanding of how we can apply games to teach mathematics.

Keywords: graph theory, game-based learning, mathematical education, qualitative methods, learner engagement

1. Introduction

In universities around the world, mathematical courses are usually taught in a 'definition-theorem-proof' format [1]. In The Mathematical Experience, Davis and Hersh assert that 'a typical lecture in advanced mathematics … consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation' [2]. The definition-theorem-proof format consists of a professor providing a lecture to students while the students take notes and try to absorb whatever the professor is saying. The lectures also follow a certain order. The main objective of the lectures is to ensure that students are able to provide rigorous proofs to the mathematical concepts that were taught [1].

The prevalence of the definition-theorem-proof format in mathematical education in University can be attributed to the primary goal of a professor—to publish research. Coming up with more engaging methods to teach the mathematical concepts is

time-consuming [3]. Without any motivation for a change in teaching methodology, this leads to mathematical courses continuing to be taught in such formats.

Although the definition-theorem-proof format has been widely used in universities across the world, it has been frowned upon by many mathematicians and math educators. Researchers argue that the definition-theorem-proof format scares students [3, 4], which may lead them to have lower confidence and thus lower self-efficacy [5], and merely provides the students the finished product in a proof. That is, students do not see what goes behind the scenes to arrive at certain steps in a proof [2, 6]. This leads to students not being able to think for themselves when it comes to problem-solving. A good example would be epsilon-delta proofs for limits in Calculus. The definition-proof-theorem format often does not offer much motivation into why some mathematical objects are the way they are and thus does not offer much meaningful learning for the students [7]. In addition, students who are taught with this format learn less than we would like them to learn due to the passive nature of learning that is brought about by the definition-theorem-proof format [7].

1.1 Why is game-based learning useful when learning mathematics?

We first define game-based learning, and we take this definition from a research paper: 'Game-based learning encourages active learning and engagement by providing students with possibilities to place problem-solving within the context of play' [8]. There are a plethora of reasons why game-based learning is beneficial for learning mathematics. Games are able to engage students [9], allow for students to investigate mathematical concepts from the games [10], and also promote discussion about the mathematical concepts [11]. Vankúš's [8] systematic review of 58 papers on the impact of game-based learning in mathematics education found that most studies (84%) indicated positive influences on student motivation and attitudes towards mathematics.

Game-based learning has been gaining traction in the recent years, and we can see attempts were made in the past to use games to teach concepts in graph theory. In a research paper, a board game was used to introduce computer science students to the basics of graph theory [12]. This may lead one to think that game-based learning may only be appreciated by a more mature audience. However, there were two studies that used games to teach shortest path algorithms to children aged 5–17 [13] as well as to university students [14].

1.2 Introduction to graph theory

In this section, we introduce the notion of a graph to the readers (**Figure 1**).

Definition 1 (Graph) A graph G = (V, E) consists of a nonempty set V of *vertices* and a set E of *edges*. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints [15].

We can also refer to a graph as a collection of circles (vertices) and lines (edges).

Definitions 2, 3, and 4 are some concepts that are in Graph Theory.

Definition 2 (K-vertex colouring) A *k-vertex colouring* of a graph G = (V,E) is a function

$$f : V \rightarrow \{1, \dots, k\}$$

such that for all $u, v \in V$ with $(u, v) \in E$, we have $f(u) \neq f(v)$. [15].

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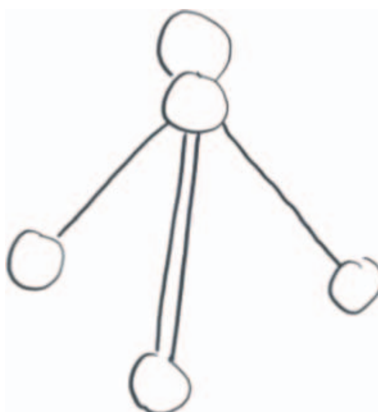

Figure 1.
Example of a graph.

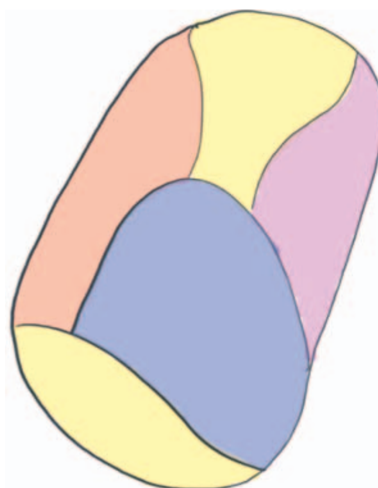

Figure 2.
Colouring of a map.

In layman terms, a colouring of the graph such that no two vertices connected by an edge have the same colour. Referring to **Figure 2**, we can also visualise it like a map; in this case, any two countries that are beside each other cannot have the same colour.

Definition 3 (Graph Isomorphism) Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic*, if there is a bijection function $f : V_1 \to V_2$, such that for all $u, v \in V_1$:

$$\{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2.$$

[15].

The definition of Isomorphism may look daunting to readers; however, in layman terms, two graphs being isomorphic just means that they have the same structure although they may look different visually.

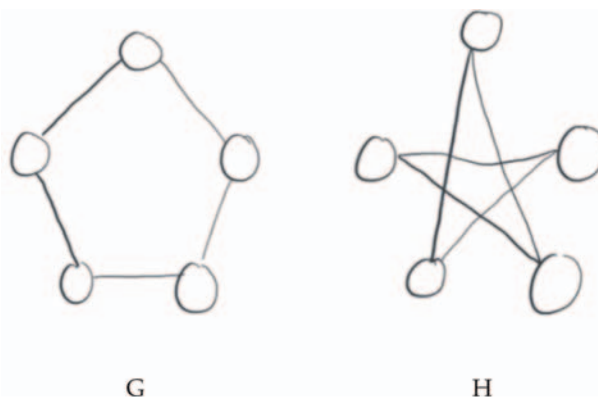

Figure 3.
Example of isomorphic graphs.

Referring to **Figure 3**, we can think of the graph G as a rubber band, and we know that we can twist a rubber band to attain a star which is H. Thus, both graph have the same structure, but we draw it differently.

Definition 4 (Planar graph) A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph [15]. The graph G in **Figure 3** is an example of a planar graph.

Another interesting concept in graph theory is the concept of Vertex Magic Total Labelling of Selected Trees.

Definition 5 (Vertex Magic Total Labelling) Let G = (V,E) be a graph such that $|V| = v$ and $|E| = e$. A vertex magic total labelling of G is a one-to-one function $f : V \to \{1,2,3, \dots, v + e\}$ that satisfy the following property:

- Let $v, v' \in V$ and $e_1, \dots, e_n \in E$ be the edges incident to v and $e'_1, \dots, e'_p \in E$ be edges incident to v'. Then $f(v) + f(e_1) + \dots + f(e_n) = f(v') + f(e'_1) + \dots + f\left(e'_p\right)$ [16]

2. Methodology

To find out, we can use games to teach graph theory concepts. We developed a series of games based on graph theory concepts such as graph coluring, graph Isomorphism, and graph planarity that were taught in the module MH1301: Discrete Mathematics that were taught to undergraduates in Nanyang Technological University(NTU) at the NTU School of Physical and Mathematical Sciences (SPMS).

A 2-hour workshop will be designed based on the games and will be conducted for students undertaking the module MH1301: Discrete Mathematics.

Rules of sudoku

- Each row, column, and 2x2 box in a 4x4 grid must contain the numbers 1–4 without repetition (**Figure 4**).
- The game ends when the entire grid is filled in correctly.

Sudoku is a game that many has been introduced to during their childhood which makes it accessible to many. This familiarity with the game allows participants to not

Figure 4.
Sudoku.

feel overwhelmed. Also, completing a sudoku puzzle can give participants a sense of accomplishment.

Thus, we attempt to use sudoku to teach the concept of graph colouring. It can be seen that in each 2x2 grid, we cannot repeat the numbers 1–4. This can be translated to a graph colouring, but instead of using colours, we are using numbers.

Rules for code breaker

- On each piece of paper, there will be two figures; both figures have the same structure, but one is orientated/twisted in another way (**Figure 5**).

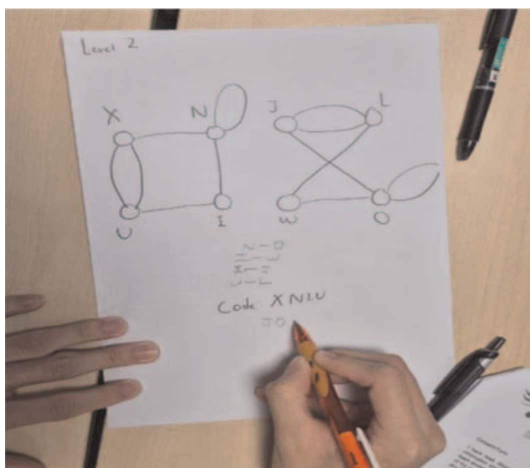

Figure 5.
Code breaker.

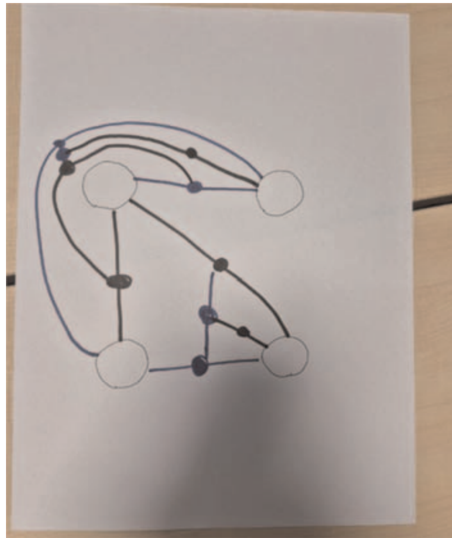

Figure 6.
Sprouts.

- Find the right orientation of the figures, and match the letter to decode the message.

The process of orientating or twisting the graphs to find a proper matching between the vertices allows us to show that the graphs are isomorphic. We added in the element of mystery that allows us to engage the participants.

How to play sprouts:

1. Players will take turns to draw a line between two spots or from a spot to itself; afterwards, they will add a new spot somewhere along the line that they drew (**Figure 6**).

2. The lines cannot cross each other.

3. No spot may have more than three lines attached to it. A line from the spot to itself counts as two attached lines.

4. The game ends when there is no available moves left.

5. The player who makes the last move wins.

Sprouts was chosen to teach the concept of Graph Planarity. Due to rule 2, when playing the game, players must be wary to not have the graph have any edges crossing. This means that the player will be dealing with planar graphs. This allows for us to link the concept of graph planarity to the participants. In playing the game, the participants will realise that planar graph will not have too many edges.

Sprouts is also a two-player game, meaning that participants will be able to play against their friends. The element of competition can help participants to be engaged in the learning process.

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A 20-minute interview (see Appendix A) to examine the effects of how the games has brought about understanding in the graph theory concepts that will be administered following the workshop. We decided to do an interview over a survey because it would allow us to probe further to gain a deeper understand of the participants experiences and perspectives during the workshop which may be difficult to achieve with the fixed response options of a survey.

3. Preliminary results and discussion

This study examined the use of games to teach mathematical concepts to students. The findings of this study will be grouped in five subsections:

3.1 Traditional methods of learning

Traditionally, when mathematical topics are first introduced to learners in a typical classroom setting, definitions and foundational concepts are first laid out for the learner to build up the prerequisite knowledge required to understand future theorems/proofs that will eventually be derived [1]. Such definitions and concepts, especially for mathematics at the undergraduate level, are algebraic in nature. For example, many engineering, mathematics, and computer science majors will most likely have to take a course in Linear Algebra where they would come across the concept of a vector space which is a generalisation of our Euclidean spaces. It can be difficult for the learner to visualise these abstract concepts when it has little to no direct relevance to real-life problems from a beginner's perspective [17].

For an undergraduate education course in mathematics, lectures are the typical mode of instruction. Some independent learning is expected from the student to bridge any knowledge gaps that may have surfaced during the lecture. They may choose to opt for online resources (such as videos or articles) or offline materials (such as the course's textbook, or other related books).

One may question the efficacy of such self-directed learning tasks when they are not scaffolded for the learner. Some learners may have had some experience in college education on knowing what mode of learning they work best with, while others may not understand how to begin bridging these gaps of knowledge.

When asked how they usually approached learning complex mathematical concepts in their courses, participants revealed that they tended to favour using visual methods to learn mathematical concepts. Participants (P1, P4) mentioned trying to visualise the mathematical concepts:

> 'If it is a concept that I cannot really understand, typically I will go to YouTube and search up the concept and try to find visual representations to better understand the concept' (P1).
>
> 'I'm more like a visual learner. So whatever they give, like the definition itself, I don't learn, I try to visualise what I am doing to get fundamental understanding. Like this is the way the thing works' (P4).

3.2 Benefit of games in math learning (in the context of graph theory)

The definitions and foundational concepts straight out from an undergraduate course may be algebraic and abstract in nature, and learners may not fully grasp them

initially. Games can help with this aspect by providing a visual representation to fully cement the learner's understanding of the topic.

The games used in this study provide a visual representation of graph theory concepts for students. Participants (P1, P3) reported that being able to physically see representations of complicated concepts, such as graph Isomorphism, promoted their understanding of it.

> 'Isomorphism became a lot clearer. Previously I only knew isomorphic by the English words but being able to see it visually it definitely did help my understanding of isomorphic. The example was I think we were looking at the edges and the vertices being laid out in different ways. So sometimes they were being crossed but they actually they are the same' (P1).
> 'It helps for the isomorphism part to visualise better for the twists and turns' (P3).

It was also observed that participants started redrawing the diagrams without prompting, in order to play the game, Code Breaker, better.

The games allow for an alternate but relevant perspective on the topic at hand. By breaking down multiple complicated concepts and distilling it into various rules that the learner can follow, it effectively helps the learner bridge the gap between the practical and the abstract, and also allows them to better grasp the concepts at hand.

The games used in the study helps to simplify complicated concepts. Participants (P2,P3) mention how the games have helped to simplify complicated concepts from graph theory:

> 'The games simplify complicated concepts so it helps me understand the application and how it works better' (P2).
> 'The explanations are a bit simpler and creates a more lasting impression compared to just reading off from the notes' (P3).

This seems to suggest that playing the games may have provided participants a stress-free learning environment. Games may allow students to focus on understanding the concepts in terms of the games rules, in order to apply when playing, rather than for academic purposes.

The games chosen provide immediate short-term goals and provide the learner with a sense of achievement, motivating the learner to continue. This near-immediate form of assessment provides the learner with a personalised avenue for reflection on concepts that were applied correctly (or incorrectly).

The games used in this study are more engaging and thus capture interests in the concepts. Participants (P2, P3) talk about how playing games is more engaging and makes it easier to remember new content:

> 'Playing games is more engaging because I have a short attention span so it helps to keep my attention span and it is more memorable so I will understand and internalise the content more' (P2).
> 'I guess it is more fun and easier to absorb new knowledge' (P3).

The responses seem to suggest that the hands-on nature of games captures the attention of participants and allows them to better internalise the concepts because they have directly experienced it.

3.3 Initial perceptions of graph theory

For an undergraduate lecture setting, there may be insufficient scaffolding in place to allow the learner to have a good grasp of the concepts taught during the class. This causes the learner to lose track of their goals and may end up feeling disorientated and discouraged. When the intrinsic motivation of the student to learn has been negatively affected, the student would be less receptive to learning in the future, and this inevitably would cause a negative feedback loop.

For learners who may have had experienced such confusion in past courses, they may be hesitant when approaching new topics in other courses as they may want to prevent such a negative scenario from playing out again.

Graph theory seems difficult to those who have not learn it before. Participants (P2, P3) reported feeling unconfident and overwhelmed at the prospect of learning graph theory:

> 'For me before the workshop I felt that graph theory daunting and very scary. But I feel like after the workshop I think it is very useful in a certain way and it is very interesting and I would love to learn more about it' (P2).
> 'My confidence also increase after the workshop cos before the workshop we were quite stressed whether we could even understand it but I think through this workshop we can visualise things more' (P5).

This may be attributed to the technical jargon used in graph theory theorems and their definitions.

3.4 Changes in perceptions towards graph theory after game-based learning

By linking relatable and relevant concepts from the lectures to the games being played, learners are exposed to direct applications of the purely symbolic content, helping them make connections to the bigger picture behind the topic(s).

The games used in the study capture interest *via* showing real world applications. Participants (P1, P2) reflected that being able to understand how graph theory can be applied in the real world piqued their interest in understanding the theory behind the games.

> 'I think for me the one thing that was interesting was the applications that we can use graph theory on. I think there are quite a few problems and I think graph theory can help understand and solve those type of problems. I think Merrill did share about networking stuff which I think for me im quite interested in. I am slightly more interested in knowing how graph theory works'.
> '[…] I feel like after the workshop I think it is very useful in a certain way and it is very interesting and I would love to learn more about it'.

The games were developed from some form of application of graph theory. This may have in turn brought about interest regarding the applications of graph theory in the real world.

3.5 Changes in self-efficacy beliefs towards learning graph theory

The games act as a stress-free feedback diagnostic tool for the students to assess their own learning, helping them view their own progress in

terms of understanding the content would push them to become more motivated learners.

The understanding enhanced by games used in the study increased the participants' confidence in their ability to learn graph theory.

> 'I guess the confidence level got increase by abit cos like I feel like i understand the concept abit better compared to before' (P3).
> 'My confidence also increase after the workshop cos before the workshop we were quite stressed whether we could even understand it but I think through this workshop we can visualise things more' (P5).

It can be seen that teaching methods such as gamification can promote understanding which leads to increase feelings of self-efficacy.

4. Conclusions

The main goal of our research is to learn how we can use games to teach mathematics to students. Our findings that the games used in the study manages to engage the students in learning Graph theory supports existing literature which posits that games are able to engage students [9]. Our findings also show that the games used in the study are able to capture their interest which supports Vankúš's [8] review of 58 papers that most studies indicated positive influences on student motivation and attitudes towards mathematics. A limitation for the research is that although we collected data on how the games were useful in their understanding in the concepts, we did not collect any data for how this understanding managed to help them with their tests and finals. For future work, we would like to further push game-based learning into other branches of mathematics such as abstract algebra, calculus or linear algebra and also to expand to other graph theory concepts such as vertex magic total labelling (definition 5). It would also be interesting to explore whether digital games or non-digital games would be more efficient to teach mathematics to students. It may also be useful to find out if game-based learning for mathematical concepts can lead to high test scores.

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A. Appendix A

Here we list the interview questions that we asked the participants.

1. Can you describe your overall experience with learning Graph Theory prior to this workshop? What were some of the challenges you faced?

2. How do you typically approach learning complex mathematical concepts in your courses?

3. In what ways did the game presented in the workshop help you understand the graph theory concepts better?

4. Were there specific aspects of the mathematical concepts that became clearer to you through the use of games? Can you give an example?

5. How does the experience of learning with games compare to more traditional methods you have encountered in your coursework?

6. Has your confidence in solving mathematical problems within Graph Theory changed after attending the workshop? Please elaborate.

7. Did the workshop influence your interest or engagement with Graph Theory? In what way?

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